Consumer Gradual Learning and Firm Non-stationary Pricing

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Abstract

Recent privacy regulations have disrupted firms' ability to track individual consumers in real time. Without the ability to tailor prices to evolving consumer beliefs, the only stationary pricing strategy is a constant price. This paper introduces a novel framework where firms adopt *non-stationary pricing strategies* – prices that evolve over time without being contingent on consumers' current valuation. We show that non-stationary pricing strategies can outperform stationary ones. Under zero search costs, a perfectly patient firm's optimal price is arbitrarily close to constant, but with discounting, the slope of the optimal price is bounded from zero. When search costs are positive, the optimal price is non-stationary even if the firm is perfectly patient. The price increases over time if the information is too noisy or the search cost is too high. The direction of price trajectories is more nuanced in other cases where consumers have a stronger incentive to search, with increasing prices for consumers with high or low initial valuation and decreasing prices for medium–value consumers.

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1 Introduction

Consumers often gather information gradually to reduce uncertainty about a product's value before making a purchasing decision. They might visit a seller's website, read reviews on the retailer's storefront, or search for review articles through search engines, yet these activities only partially resolve their uncertainty. Building on the seminal work of Weitzman (1979) and Wolinsky (1986), a rich literature has explored optimal consumer search strategies across multiple alternatives or product attributes (Moscarini and Smith, 2001; Armstrong et al., 2009; Branco et al., 2012; Ke and Villas-Boas, 2019; Zhong, 2022; Chaimanowong et al., 2023) and their implications for firms' strategies in information provision, pricing, product design, and advertising (Anderson and Renault, 2006; Villas-Boas, 2009; Bar-Isaac et al., 2010; Mayzlin and Shin, 2011; Chaimanowong and Ke, 2022; Ke et al., 2023; Yao, 2023). A critical dimension of these studies is pricing, with existing work often assuming exogenous prices, endogenous constant prices, or endogenous prices contingent on the consumers' current valuation. However, recent privacy regulations like GDPR and CCPA have disrupted firms' ability to track individual consumers in real time, making it challenging to tailor prices to evolving consumer beliefs. Even if a firm can track consumers' browsing behavior, it may be hard for the firm to know how consumers will interpret the information they see. For example, Tesla may be able to observe that a consumer clicks on an image of the interior design of the car but may not know whether the consumer likes the large screen on Tesla or the traditional dashboard. This calls into question whether the firm can track the consumer's belief evolution process when the consumer is searching for information. This raises an important question: can firms benefit from dynamic pricing when consumer belief evolution is unobservable?

Without the ability to track consumers' valuation evolution, the only stationary pricing strategy is a constant price. This paper introduces a novel framework where firms adopt *non-stationary pricing strategies* – prices that evolve over time without being contingent on consumers' current valuation. We address two key questions: (1) Is a stationary pricing strategy always optimal for a firm that cannot observe belief evolution? (2) If not, what are the characteristics of the optimal non-stationary pricing strategy?

Our findings challenge the conventional reliance on stationary pricing. We show that non-stationary pricing strategies can outperform stationary ones. We prove that a consumer can do almost as well by approximating any price which is sufficiently slow-moving by linear price if she is sufficiently myopic, which can be a building block for future research to simplify the strategy space of non-Markov problems. Given this result, by assuming that the consumer is sufficiently myopic and the price is linear and varies slowly, we show that, under zero search costs, a perfectly patient firm's optimal price is arbitrarily close to constant, but with discounting, the slope of the optimal price is bounded from zero. When search costs are positive, the optimal price is non-stationary even if the firm is perfectly patient. The price increases over time if the information is too noisy or the search cost is too high. The direction of price trajector is more nuanced in other cases where consumers have a stronger incentive to search, with increasing prices for consumers with high or low initial valuation and decreasing prices for medium–value consumers.

Our contributions are twofold. First, we provide a theoretical advance in optimal control by incorporating non-stationary strategies into a consumer search framework. Unlike most economic models, which impose stationarity for tractability, our results highlight that such restrictions may lead to sub-optimal outcomes. While a few earlier papers have explored non-stationarity in search problems driven by exogenous environment such as the finite horizon and the evolving distribution of rewards (Gilbert and Mosteller, 1966; Sakaguchi, 1978; Van den Berg, 1990; Smith, 1999; Kamada and Muto, 2015), we model endogenous nonstationarity arising from firms' strategic pricing in response to consumer gradual learning. To our knowledge, this is the first paper to study endogenous non-stationary pricing under consumer gradual learning, providing a foundational step toward understanding firms' nonstationary interventions in this context.

Second, we offer practical insights into how firms can adapt to privacy regulations. Nonstationary pricing leverages time – a freely available and regulation resistant resource – as an information source for pricing decisions, reducing reliance on costly tracking technologies. This is especially relevant in light of privacy-driven disruptions, such as Apple's iPhone privacy upgrades, which cost publishers nearly \$10 billion in ad revenue in 2021 alone.¹ Privacy regulations can prevent firms from tracking consumers' demographic information, browsing behavior, and other characteristics, but cannot ban the time to which everyone has access. While existing studies focus on the economic impact of privacy regulations (Goldfarb and Tucker, 2011; Athey et al., 2017; Montes et al., 2019; Ichihashi, 2020; Baik and Larson, 2023; Ke and Sudhir, 2023; Ning et al., 2023; Bondi et al., 2023; Goldberg et al., 2024; Yao, 2024), our work explores how firms can proactively adjust their pricing strategies to thrive in a privacy-conscious environment.

¹ Source: https://www.businessinsider.com/apple-iphone-privacy-facebook-youtube-twitter-snap-lose-10-billion-2021-11.

2 The Model

A firm offers a product with a marginal cost of g and chooses the price. A consumer decides whether to purchase it or not. The consumer's initial valuation is v_0 , which is common knowledge. The initial valuation represents the consumer's knowledge about the product based on past experiences, word of mouth, or advertising. Before making a decision, she can gradually learn about various product attributes to update her belief about the product's value. We will focus on the learning processes arises from the general non-linear optimal filtering framework (Liptser and Shiryaev, 2013, Chapter 8). Suppose that the total utility the consumer gets from consuming the product is given by an unobservable process $\{\pi_t\}_{t\geq 0}$. The consumer pays the flow search cost cdt per dt time to learn about π_t by observing a process $\{S_t\}_{t\geq 0}$ which generates a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ such that $v_t := \mathbb{E}[\pi_t | \mathscr{F}_t]$ is a square-integrable continuous martingale². We will some times write the valuation process as $\{v_s^{t,x}\}_{s\geq t}$ if we would like to emphasize the initial condition $v_t^{t,x} = x$, or as $\{v_t^x\}_{t\geq 0}$ if the initial condition $v_0^x = x$ is at t = 0, or simply as v_t when the initial value x is not so important. We shall assume that $\{v_s^{t,x}\}_{s\geq t}$ is the unique strong solution to:

$$dv_s^{t,x} = \mu(v_s^{t,x}, \pi_s)ds + \sigma(v_s^{t,x})dW_s, \qquad v_t^{t,x} = x,$$
(1)

where $\{W_t\}_{t\in\mathbb{R}_{\geq 0}}$ is the standard Brownian motion adapted to the filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in\mathbb{R}_{\geq 0}}, \mathbb{P})$. In particular, we have $\mathbb{E}[dv_t|\mathscr{F}_t] = 0$ and $\mathbb{E}[(dv_t)^2|\mathscr{F}_t] = \sigma(v_t)^2 dt$. Additionally, we assume $\mu(.)$ and $\sigma(.)$ to be smooth and satisfies the global Lipschitz condition:

$$|\mu(x,\pi_t) - \mu(y,\pi_t)| + |\sigma(x) - \sigma(y)| \le L|x-y|, \quad \text{for all } t, x, y \in \mathbb{R}$$
(2)

for some constant $L \ge 0$. Given $\mathbf{x} = (t, x) \in \mathbb{R}^2$, we shall write $\mu(\mathbf{x})$ and $\sigma(\mathbf{x})$ instead of $\mu(x)$ and $\sigma(x)$ when it is more convenience to work in the 'vector' notation, even though $\mu(.)$ and $\sigma(.)$ do not depend directly on t.

Previous works that study the firm's endogenous pricing strategy in the presence of consumer gradual learning either assume that the firm perfectly observes the consumer's valuation evolution processes (v_t is common knowledge between the consumer and the firm) and can condition the price on the consumer's current valuation, or that the firm charges a constant price over time. Suppose we define the state variable by the consumer's current valuation, as is standard in the literature. The firm's problem in the first scenario is to choose the optimal *Markov strategy*. This setup does not always fit real-world examples. Recent

² A sufficient condition for $\{v_t\}_{t\geq 0}$ to be martingale is if $\{\pi_t\}_{t\geq 0}$ is martingale. The term squareintegrability refers to the condition $\mathbb{E}[v_t^2] < \infty$ for all $t \geq 0$.

privacy regulations like GDPR and CCPA have disrupted firms' ability to track individual consumers in real time. Even if a firm can track consumers' browsing behavior, it may be hard for the firm to know how consumers will interpret the information they see. Moreover, in many offline settings, individual-level tracking is not feasible.

Without the ability to observe the consumer's valuation evolution and tailor prices based on v_t , the only stationary pricing strategy is a constant price. Is a stationary pricing strategy always optimal for a firm in such cases? The major innovation of this paper is to consider *non-stationary pricing strategies*, prices that evolve over time without being contingent on consumers' current valuation. Formally, the firm can commit to a pricing scheme p := $\{p_t\}_{t\geq 0} \in \mathcal{P}$, where \mathcal{P} is a subset of smooth functions on $[0,\infty)$, $\mathcal{P} \subset C^{\infty}[0,\infty)$. This pricing strategy is a *non-Markov strategy* because p_t depends on history (time t) other than the current state v_t . It is widely known in optimal control that it is much harder to characterize *non-Markov strategies* than *Markov strategies*.

The consumer's search strategy consists of choosing an appropriate stopping time. We denote by \mathcal{T} the set of all stopping times adapted to $\{\mathscr{F}_t\}_{t\in\mathbb{R}_{\geq 0}}$. We formalize the setup as a game with two players, a consumer ("Buyer" B) and a firm ("Seller" S), playing in the following sequence:

- 1. At t = 0, the firm commits to a pricing strategy $p \in \mathcal{P} \subset C^{\infty}[0, \infty)$.
- 2. At any t > 0, the consumer decides whether to purchase the product, exit, or search for more information.
- 3. The game ends when the consumer makes a purchase or exits.

The only knowledge the seller has about the consumer is their initial valuation, v_0 , which may be derived from a survey conducted over a large population. Importantly, when the consumer decides whether to purchase the product, exit, or keep searching at any given time, she takes into account both the current price and the future price trajectory. For any $p \in \mathcal{P}$ and $\tau \in \mathcal{T}$, we define ³

$$\mathcal{V}^{B}(t,x;\tau,p) := \mathbb{E}\left[e^{-r(\tau-t)}\max\{v_{\tau} - p_{\tau}, 0\} - \int_{t}^{\tau} c e^{-r(s-t)} ds \mid v_{t} = x\right]$$
(3)

and

$$\mathcal{V}^{S}(x;\tau,p) := \mathbb{E}\left[e^{-m\tau}(p_{\tau}-g) \cdot \mathbf{1}_{v_{\tau} \ge p_{\tau}} \mid v_{0} = x\right]$$

$$\tag{4}$$

as the corresponding consumer's, and the firm's expected payoffs, respectively.

³ For simplicity, we use p to denote $\{p_t\}_{t\geq 0}$ whenever this does not cause confusion.

Commitment Assumption

We assume that the firm has dynamic commitment power. It would be a relatively strong assumption if the firm could track consumer valuation processes due to the hold-up problem. When new information (consumer's current valuation) arrives, the firm has an incentive to deviate from the pricing scheme announced at the beginning to extract more surplus from the consumer. Anticipating the firm's incentive to deviate, the consumer will not start searching without a commitment device. Ning (2021) shows that a firm without commitment power can address the hold-up problem by offering a "list price." In contrast, the firm does not need to offer a list price if it has dynamic commitment power. In such cases, whether the firm has commitment power will lead to qualitatively different results.

In our setting, the firm does not receive new information about consumer valuation v_t over time. In addition, the only new information the firm can learn at time t is whether the consumer has made a purchasing decision, which will not affect the firm's strategy because the game ends whenever the consumer purchases the product or exits. The firm does not have any new information during the game. Therefore, there is no hold-up problem and the firm does not have an incentive to deviate from the announced pricing strategy. Thus, the commitment assumption does not qualitatively affect the equilibrium outcome in this case. We make the assumption mainly for a cleaner analysis and presentation.

Solution Concept

We consider the following equilibrium concept.

Definition 1. An ε -Subgame perfect Nash's equilibrium (ε -SPNE) consists of:

$$(\{\tau^*[p] \in \mathcal{T}\}_{p \in \mathcal{P}}, p^* \in \mathcal{P})$$

such that: for all $p \in \mathcal{P}$,

$$\mathcal{V}^{B}(t, x; \tau^{*}[p], p) \geq \mathcal{V}^{B}(t, x; \tau, p) - \varepsilon, \quad \forall \tau \in \mathcal{T},$$

and
$$\mathcal{V}^{S}(x; \tau^{*}[p^{*}], p^{*}) \geq \mathcal{V}^{S}(x; \tau^{*}[p], p) - \varepsilon, \quad \forall p \in \mathcal{P}.$$

The consumer's value function given the seller's pricing strategy p is:

$$V^B(t,x;p) := \sup_{\tau \in \mathcal{T}} \mathcal{V}^B(t,x;\tau,p).$$
(5)

When there is no ambiguity, we will compactly write $V^B(t,x) = V^B(t,x;p)$. Analogously,

we define the seller's value function:

$$V^{S}(x) := \sup_{p \in \mathcal{P}} \mathcal{V}^{S}(x; \tau^{*}[p], p)$$
(6)

Our choice of the equilibrium concept is motivated by the greater analytical traceability of the problem via perturbation theory to the order of ε . For instance, we later solve for an analytical closed form of a myopic consumer's strategy to a slow moving pricing using linear approximation to the order of ε . For further discussion we refer to Assumption 1. There is also a technical reason for such an equilibrium concept as we do not need to be concerned about the existence of $\tau^*[p] \in \mathcal{T}$ or $p^* \in \mathcal{P}$ that achieve the supremum. In a certain case, it is possible to show that the firm's profit supremum can only be approached via a limit of an admissible pricing strategy (we did not require \mathcal{P} to be closed in general).

3 Consumer's Strategy

The consumer faces an optimal stopping problem. She needs to determine the purchasing and quitting boundaries at any time. When the price is non-stationary, the consumer's purchasing and quitting boundaries are also time-contingent. This time-varying property makes her optimal stopping problem challenging even if we fix a pricing scheme. To illustrate the impact of non-stationary pricing on the consumer's problem, we first review the benchmark with constant price.

3.1 A Stationary Pricing Benchmark

When the price is constant, $p_t = p_0 \in \mathbb{R}$, the consumer's search strategy does not depend on time. In particular, we have a time-independent value function $V^B(t, x; p_0) = V_0^B(x; p_0)$, the purchasing threshold $\bar{V}_t = p_0 + \bar{V}$, and the quitting threshold $\underline{V}_t = p_0 + \underline{V}$. The value function of the consumer satisfies the Hamilton–Jacobi–Bellman (HJB) equation:

$$\frac{\sigma(x)^2}{2}\partial_x^2 V_0^B - rV_0^B - c = 0$$
(7)

subjects to the value matching condition and smooth pasting conditions:

$$V_0^B(p_0 + \bar{V}; p_0) = (p_0 + \bar{V}) - p_0, \quad \partial_x V_0^B(p_0 + \bar{V}; p_0) = 1,$$
$$V_0^B(p_0 + \underline{V}; p_0) = 0, \quad \partial_x V_0^B(p_0 + \underline{V}; p_0) = 0.$$

We refer to Strulovici and Szydlowski (2015) for the derivation of the free-boundary ODE problem from the optimal stopping problem⁴ along with the results which guarantees the existence and uniqueness of the solution in our setting. In particular, we find V_0^B to be continuously differentiable for all $x \in \mathbb{R}$, and twice continuously differentiable for all $x \in \mathbb{R}$, $\mathbb{R} \setminus \{p_0 + \underline{V}, p_0 + \overline{V}\}$.

3.1.1 Learning product attributes

We consider the learning process studied in Branco et al. (2012). The consumer gradually learns about various product attributes to update her belief about the product's value before making a purchase decision. We assume each attribute *i* has a ground-truth utility of x_i relative to the outside-option counterpart. The total product's utility relative to the outside option (which we normalized to zero) based on the *t* searched attributes is $\pi_t := \sum_{i=0}^t x_i$. When there are infinitely many attributes, each with a very small weight in value, π_t becomes a Brownian motion: $d\pi_t = \sigma dW_t^{\pi}$. The consumer learns about $\{\pi_t\}_{t\geq 0}$ by observing the signal $\{S_t\}_{t\geq 0}$, where $dS_t := \pi_t dt + \sigma_S dW_t$, for some σ_S representing the information quality, or the amount of attention the consumer gives in learning. Assume the normal prior belief $\pi_0 \sim \mathcal{N}(v_0, \sigma \sigma_S)$, then we have $dv_t = \frac{\sigma}{\sigma_S}(\pi_t - v_t)dt + \sigma dW_t$, or simply:

$$dv_t = \sigma dW_t^v \tag{8}$$

by Lévy characterization, where $\{W_t^v\}_{t\geq 0}$ is a standard Brownian motion adapted to $\{\mathscr{F}_t\}_{t\geq 0}$. We shall write W_t^v simply as W_t for convenience, hereafter. Figure 1 illustrates the evolution of the consumer's valuation as the consumer checks more and more attributes. The stationary structure leads to closed-form solutions as given in Branco et al. (2012):

$$V_0^B(x;p_0) = \frac{c}{r} \left[\cosh \frac{\sqrt{2r}}{\sigma} \left(x - \underline{V} - p_0 \right) - 1 \right], \tag{9}$$

$$\bar{V} := \sqrt{\frac{c^2}{r^2} + \frac{\sigma^2}{2r} - \frac{c}{r}},$$
(10)

$$\underline{V} := \left(\sqrt{\frac{c^2}{r^2} + \frac{\sigma^2}{2r}} - \frac{c}{r}\right) - \frac{\sigma}{\sqrt{2r}}\log\left(\sqrt{\frac{r\sigma^2}{2c^2}} + \sqrt{1 + \frac{r\sigma^2}{2c^2}}\right). \tag{11}$$

⁴ Unlike in Strulovici and Szydlowski (2015) our $\mu(.)$ also depends on another unobservable process $\{\pi_t\}_{t\geq 0}$. One work–around is to consider the process $X_t := (v_t, \pi_t)$ which generates the full information filtration $\{\mathscr{F}_t^X\}_{t\geq 0}$, follows Strulovici and Szydlowski (2015), then take another expectation condition on $\{\mathscr{F}_t\}_{t\geq 0}$ at the end. Such detail is not so critical in this section as we will merely consider a few benchmark examples for motivation and we will later study the general version of this problem rigorously via the viscosity solutions framework.



Figure 1: A sample path of the consumer's valuation evolving processes during search

3.1.2 Binary classification

We consider a product with the ground-truth value given by the time-independent binary random variable $\pi_t = \pi \in \{0, 1\}$. To make a purchase decision, the consumer must classify whether the product has a *high* value ($\pi = 1$) or *low* value ($\pi = 0$). Given the initial expectation is $v_0 = \mathbb{E}[\pi|\mathscr{F}_0] \in [0, 1]$, the consumer can further learn the value of π by observing the signal $\{S_t\}_{t\geq 0}$, where $dS_t := \pi dt + \sigma dW_t$. Then we have

$$dv_t = \frac{v_t(1 - v_t)}{\sigma^2} \left[(\pi - v_t) dt + \sigma dW_t \right].$$
 (12)

The resulting free-boundary ODE problem has been considered in Ke and Villas-Boas (2019) in the non-discounting case: r = 0. We present the solution as follows:

$$V_0^B(x;p_0) = A_+ x^{m_+} (1-x)^{m_-} + A_- x^{m_-} (1-x)^{m_+} - \frac{c}{r}$$
(13)

where $m_{\pm} := \frac{1 \pm \sqrt{1 + 8r\sigma^2}}{2}$, $A_{\pm} := \frac{(1 - m_{\mp})(\bar{V} + p_0) + (c/r - p_0)(\bar{V} + p_0 - m_{\mp})}{(m_{\pm} - m_{\mp})(\bar{V} + p_0)^{m_{\pm}}(1 - p_0 - \bar{V})^{m_{\mp}}}$, and \bar{V} , \underline{V} can be solved from:

$$\frac{(c/r - p_0)(\bar{V} + p_0 - m_{\mp}) + (1 - m_{\mp})(\bar{V} + p_0)}{(\bar{V} + p_0)^{m_{\pm}}(1 - p_0 - \bar{V})^{m_{\mp}}} = \frac{(c/r - p_0)(\underline{V} + p_0 - m_{\mp})}{(\underline{V} + p_0)^{m_{\pm}}(1 - p_0 - \underline{V})^{m_{\mp}}}.$$
 (14)

Comparing the benchmarks and our problem, we can see that stationarity simplifies the problem significantly. In the benchmark model, the consumer's entire optimal stopping strategy can be summarized by **two unknowns**: \bar{V} and \underline{V} . The consumer will purchase

the product at any time during the search if her valuation reaches the purchasing threshold and will quit searching at any time if her valuation reaches the quitting threshold. In contrast, the consumer's entire optimal stopping strategy consists of **an infinite number of unknowns**. Knowing that the price changes over time, the consumer's purchasing and quitting thresholds also evolve. She has different purchasing and quitting thresholds at different times. So, instead of pinning down a one-dimensional purchasing/quitting threshold, we need to determine a two-dimensional purchasing/quitting boundary. These time-dependent thresholds significantly complicate our problem.

3.2 Consumer's Strategy under Non–Stationary Pricing

Let us first consider the subset \mathcal{P}_T of all the admissible pricing strategies $\mathcal{P} \subset C^{\infty}[0,\infty)$ given by the strategies that are constant after some amount of time T > 0:

$$\mathcal{P}_T := \{ p \in \mathcal{P} \mid p_t = p_T, \forall t \ge T \}.$$

We consider when T > 0 is large but finite to provide the boundary condition at t = Tneeded for the existence and uniqueness result, but in many cases, there will be no problem taking the limit $T \to \infty$. We start with the following intuitive characterization of $V^B(t, x; p)$:

Lemma 1. Let $p \in \mathcal{P}_T$ be a pricing strategy, then $V^B(t, x; p)$ is monotonically increasing in x for any fixed t.

Proof. Consider any $x, x' \in \mathbb{R}$, and suppose that x' > x. Let $\{v_s^{t,x}\}_{s \ge t}$ and $\{v_s^{t,x'}\}_{s \ge t}$ be the two strong solutions of the SDE (1), and we have $v_s^{t,x'} > v_s^{t,x}$ a.e., for all $s \ge t$. This can be seen by using the Lipschitz condition (2) to analyze the difference process $d_s := v_s^{t,x'} - v_s^{t,x}$. It follows that $\mathcal{V}^B(t,x';\tau,p) \ge \mathcal{V}^B(t,x;\tau,p)$ for all $\tau \in \mathcal{T}$, hence $V^B(t,x') \ge V^B(t,x)$. \Box

Instead of directly finding the optimal $\tau^*[p] \in \mathcal{T}$ to the optimization problem (5) we consider the corresponding HJB equation:

$$H(t, x, V, \nabla V, \Delta V) = 0 \tag{15}$$

where $H: \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \times \mathcal{S}_2(\mathbb{R}) \to \mathbb{R}$ is given by

$$H\left(t, x, V, \nabla V, \Delta V\right) := \min\left\{c + rV - \partial_t V - \frac{\sigma(x)^2}{2}\partial_x^2 V, V - x + p_t, V\right\},\$$

with $S_2(\mathbb{R})$ denoted the space of 2 × 2 symmetric matrices. Since $p \in \mathcal{P}_T$ is only defined for $t \ge 0$, to discuss the solution on \mathbb{R}^2 let us extends it by defining $p_t = p_0$ for all t < 0, analogous to how we defined $p_t = p_T$ for t > T. We consider the following asymptotic boundary conditions:

$$\lim_{t \to +\infty} V(t,x) = V_0^B(x;p_T), \qquad \lim_{t \to -\infty} V(t,x) = V_0^B(x;p_0)$$

$$V(t,x) = x - p_t, \ \forall x \ge \bar{V}_t[p], \qquad V(t,x) = 0, \ \forall x \le \underline{V}_t[p]$$
(16)

for some functions $\overline{V}[p], \underline{V}[p] : \mathbb{R} \to \mathbb{R}$, depending on $p \in \mathcal{P}_T$ and $\overline{V}_t[p] \ge \underline{V}_t[p], \forall t \in \mathbb{R}$.

Remark 1. We argue informally as follows that the value function V^B satisfies the boundary condition (16). For $t \ll 0$ or $t \gg T$ the consumer treat the search problem as if the price is constant at p_0 or $t = p_T$, respectively. For all sufficiently high $x \gg \max_{t \in [0,T]} |p_t|$, it is best to immediately purchase and receive the expected payoff $x - p_t$. Waiting for any extra $\delta > 0$ amount of time means the expected utility x will be discounted by $e^{-r\delta}$ which is greater in absolute terms than any gains from price decreases. On the other hand, consider any sufficiently low $x \ll \min_{t \in [0,T]} |p_t|$, it is best to immediately exit and receive zero payoff. In this case, the probability the process v will overtake p in the near future is small and it is out-weighted by the search cost.

The purchase and exit boundaries: $\overline{V}[p]$, and $\underline{V}[p]$, provide a simple characterization of the learning strategy. The following are some observations of their behavior.

Proposition 1. Let $p \in \mathcal{P}_T$ be a pricing strategy and $h : \mathbb{R} \to \mathbb{R}_{\geq 0} \in \mathcal{P}_T$ be strictly monotonically increasing over [0, T).

- 1. Suppose that $h_0 = 0$, then at t = 0 the purchase and exit boundaries under the pricing strategy $\tilde{p} := p + h$ satisfies $\bar{V}_0[\tilde{p}] \leq \bar{V}_0[p]$, and $\underline{V}_0[\tilde{p}] \geq \underline{V}_0[p]$.
- 2. Let $K \in \mathbb{R}$ be a constant, and $\bar{v}, \underline{v} \in \mathbb{R} \cup \{\pm \infty\}$ be such that $\underline{v} \leq v_t \leq \bar{v}$ a.e., for all $t \in \mathbb{R}$. Then under the pricing strategy $\tilde{p} := p + Kh$ we have:

$$\begin{split} \bar{V}_t[\tilde{p}] \searrow \max\{\tilde{p}_t, \underline{v}\}, \underline{V}_t[\tilde{p}] \nearrow \min\{\tilde{p}_t, \bar{v}\} & as \ K \to +\infty \\ \bar{V}_t[\tilde{p}] \nearrow \bar{v}, \underline{V}_t[\tilde{p}] \searrow \underline{v} & as \ K \to -\infty \end{split}$$

pointwise, for any given $t \in [0, T)$.

Remark 2. The first part of Proposition 1 also implies that if $\tilde{p} := p + h$ and $h : \mathbb{R} \to \mathbb{R}_{\leq 0} \in \mathcal{P}_T$ is strictly monotonically decreasing then $\bar{V}_0[\tilde{p}] \geq \bar{V}_0[p]$, and $\underline{V}_0[\tilde{p}] \leq \underline{V}_0[p]$. This can be seen by switching the role of \tilde{p} and p in the proposition's statement. We further note that the statement is specifically at t = 0 where $h_0 = 0$. It is not true that $\bar{V}_t[\tilde{p}] \leq \bar{V}_t[p]$ and

 $\underline{V}_t[\tilde{p}] \geq \underline{V}_t[p]$ for all $t \in \mathbb{R}$ given a monotonically increasing $h : \mathbb{R} \to \mathbb{R}_{\geq 0}$, for example see the explicit calculation in the case of linear pricing in Proposition 4. Intuitively, although the increasing price may have an immediate effects in forcing a high-value consumer to purchase instead of continue searching, such effects is short-term, if the price continues to increase for long enough then such a consumer will be hesitant due to the higher price.

We need to establish the existence and uniqueness of the solution to (15) subjects to the boundary condition (16). In stochastic control problems, the classical solution does not always exist. The standard approach is to work with a relaxed notion of a *viscosity* solution (see Crandall et al. 1992):

Definition 2. Let $H : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times S_n(\mathbb{R}) \to \mathbb{R}$ be a continuous function satisfying the properness condition: $H(\mathbf{x}, v, p, X) \ge H(\mathbf{x}, u, p, X)$ if $v \ge u$, and the degenerate ellipticity condition: $H(\mathbf{x}, v, p, X) \ge H(\mathbf{x}, v, p, Y)$ if $Y \ge X$.

A continuous function $v : \mathbb{R}^n \to \mathbb{R}$ is a viscosity subsolution if for any $\mathbf{x}_0 \in D$ and any twice continuously differentiable function ϕ such that \mathbf{x}_0 is a local maximum of $v - \phi$ we have $H(\mathbf{x}_0, v(\mathbf{x}_0), \nabla \phi(\mathbf{x}_0), \Delta \phi(\mathbf{x}_0)) \leq 0.$

A continuous function $v : \mathbb{R}^n \to \mathbb{R}$ is a viscosity supersolution if for any $\mathbf{x}_0 \in D$ and any twice continuously differentiable function ϕ such that \mathbf{x}_0 is a local minimum of $v - \phi$ we have $H(\mathbf{x}_0, v(\mathbf{x}_0), \nabla \phi(\mathbf{x}_0), \Delta \phi(\mathbf{x}_0)) \geq 0$.

A continuous function $v : \mathbb{R}^n \to \mathbb{R}$ is a viscosity solution if it is both a viscosity subsolution and supersolution.

Lemma 2. For a given $p \in \mathcal{P}_T$, the consumer's value function V^B is the unique viscosity solution to (15) subject to the asymptotic boundary conditions (16).

Proof. For convenience, in the following we will use $A_1(\mathbf{x}, V, \nabla V, \Delta V) := c + rV - \partial_t V - \frac{\sigma(\mathbf{x})^2}{2} \partial_x^2 V$, $A_2(\mathbf{x}, V, \nabla V, \Delta V) := V - x - p_t$, $A_3(\mathbf{x}, V, \nabla V, \Delta V) := V$, so that $H := \min_{i=1,2,3} A_i$. We divide the proof in to two parts, first we show that the viscosity solution to (15) subject to the specified boundary conditions is unique, then we show that the value function is a viscosity solution.

Part 1 (viscosity solution is unique):

We show that the viscosity solution to (15) subjects to the specified condition is unique. Although, this is mostly an application of the comparison principle (Crandall et al., 1992, Theorem 3.3), in our context the domain is unbounded, hence we layout the detail for completeness. Let $u : \mathbb{R}^2 \to \mathbb{R}$ and $v : \mathbb{R}^2 \to \mathbb{R}$ be viscosity subsolution and supersolution to (15), respectively, and suppose that $\lim_{t\to\pm\infty} (u - v) \leq 0$, $\lim_{x\to\pm\infty} (u - v) \leq 0$. We claim that $u \leq v$ everywhere on \mathbb{R}^2 . To show this let us assume the contrary that there exists $\hat{\mathbf{x}} \in \mathbb{R}^2$ such that $u(\hat{\mathbf{x}}) - v(\hat{\mathbf{x}}) = \max_{\mathbf{x} \in \mathbb{R}^2} (u(\mathbf{x}) - v(\mathbf{x})) > 0$. Consider the function: $w_{\alpha}(\mathbf{x}, \mathbf{y}) := u(\mathbf{x}) - v(\mathbf{y}) - (\alpha/2) \|\mathbf{x} - \mathbf{y}\|_2^2$ for some constant $\alpha \ge 0$. The assumption on the boundary conditions of u and v implies that for any $\alpha \ge 0$ there exists a local maximum $(\mathbf{x}_{\alpha}, \mathbf{y}_{\alpha}) \in \mathbb{R}^2 \times \mathbb{R}^2$ of w_{α} , and by (Crandall et al., 1992, Theorem 3.1):

$$\lim_{\alpha \to \infty} \alpha \|\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}\|_{2}^{2} = 0, \quad \lim_{\alpha \to \infty} \left(u(\mathbf{x}_{\alpha}) - v(\mathbf{x}_{\alpha}) - \frac{\alpha}{2} \|\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}\|_{2}^{2} \right) = u(\hat{\mathbf{x}}) - v(\hat{\mathbf{x}}).$$

By our assumption, we can find $\delta > 0$ such that $u(\mathbf{x}_{\alpha}) - v(\mathbf{y}_{\alpha}) \geq \delta$ for all $\alpha \geq 0$. We can apply (Crandall et al., 1992, Theorem 3.2) since \mathbb{R}^2 is locally compact, and we find $X, Y \in \mathcal{S}_2(\mathbb{R})$ such that

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$
(17)

with \mathbf{x}_{α} a local maximum of $u(\mathbf{x}) - \alpha(\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha})^{\mathsf{T}}(\mathbf{x} - \mathbf{x}_{\alpha}) - \frac{1}{2}(\mathbf{x} - \mathbf{x}_{\alpha})^{\mathsf{T}}X(\mathbf{x} - \mathbf{x}_{\alpha})$ and \mathbf{y}_{α} a local minimum of $v(\mathbf{y}) - \alpha(\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha})^{\mathsf{T}}(\mathbf{y} - \mathbf{y}_{\alpha}) - \frac{1}{2}(\mathbf{y} - \mathbf{y}_{\alpha})^{\mathsf{T}}Y(\mathbf{y} - \mathbf{y}_{\alpha})$. Since u and v are subsolution and supersolution, respectively, we have:

$$H(\mathbf{x}_{\alpha}, u(\mathbf{x}_{\alpha}), \alpha(\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}), X) \le 0 \le H(\mathbf{y}_{\alpha}, v(\mathbf{y}_{\alpha}), \alpha(\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}), Y).$$
(18)

From (17) we have:

$$A_{1}(\mathbf{y}_{\alpha}, v(\mathbf{y}_{\alpha}), \alpha(\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}), Y) - A_{1}(\mathbf{x}_{\alpha}, v(\mathbf{y}_{\alpha}), \alpha(\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}), X)$$

$$= \frac{\sigma(\mathbf{x}_{\alpha})^{2}}{2} X_{xx} - \frac{\sigma(\mathbf{y}_{\alpha})^{2}}{2} Y_{xx} = \left(\sigma(\mathbf{x}_{\alpha}) \quad \sigma(\mathbf{y}_{\alpha})\right) \begin{pmatrix} X_{xx} & 0\\ 0 & -Y_{xx} \end{pmatrix} \begin{pmatrix} \sigma(\mathbf{x}_{\alpha})\\ \sigma(\mathbf{y}_{\alpha}) \end{pmatrix}$$

$$\leq 3\alpha \left(\sigma(\mathbf{x}_{\alpha}) \quad \sigma(\mathbf{y}_{\alpha})\right) \begin{pmatrix} I & -I\\ -I & I \end{pmatrix} \begin{pmatrix} \sigma(\mathbf{x}_{\alpha})\\ \sigma(\mathbf{y}_{\alpha}) \end{pmatrix} = 3\alpha (\sigma(\mathbf{x}_{\alpha}) - \sigma(\mathbf{y}_{\alpha}))^{2} \leq 3\alpha L^{2} \|\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}\|_{2}^{2}$$

where we used the condition (2) for σ in the last inequality. Similarly, we can check that

$$A_{2}(\mathbf{y}_{\alpha}, v(\mathbf{y}_{\alpha}), \alpha(\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}), Y) - A_{2}(\mathbf{x}_{\alpha}, v(\mathbf{y}_{\alpha}), \alpha(\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}), X) \leq \|\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}\|_{2} + p_{t_{x}} - p_{t_{y}}$$
$$\leq \left(1 + \max_{t \in [0,T]} |p_{t}'|\right) \|\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}\|_{2},$$

and $A_2(\mathbf{y}_{\alpha}, v(\mathbf{y}_{\alpha}), \alpha(\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}), Y) - A_2(\mathbf{x}_{\alpha}, v(\mathbf{y}_{\alpha}), \alpha(\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}), X) = 0$. Let us define $\omega(r) := \max\left\{3L^2, 1 + \max_{t \in [0,T]} |p_t'|\right\} \cdot r$ and $i^* := \operatorname{argmin}_{i=1,2,3} A_i(\mathbf{x}_{\alpha}, v(\mathbf{y}_{\alpha}), \alpha(\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}), X)$, then

$$H(\mathbf{y}_{\alpha}, v(\mathbf{y}_{\alpha}), \alpha(\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}), Y) - H(\mathbf{x}_{\alpha}, v(\mathbf{y}_{\alpha}), \alpha(\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}), X)$$

$$\leq A_{i^{*}}(\mathbf{y}_{\alpha}, v(\mathbf{y}_{\alpha}), \alpha(\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}), Y) - A_{i^{*}}(\mathbf{x}_{\alpha}, v(\mathbf{y}_{\alpha}), \alpha(\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}), X)$$

$$\leq \omega \left(\alpha \|\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}\|_{2}^{2} + \|\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}\|_{2}^{2} \right).$$

Therefore,

$$0 < \min\{1, r\}\delta \le \min\{1, r\}(u(\mathbf{x}_{\alpha}) - v(\mathbf{y}_{\alpha}))$$

$$\le H(\mathbf{x}_{\alpha}, u(\mathbf{x}_{\alpha}), \alpha(\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}), X) - H(\mathbf{x}_{\alpha}, v(\mathbf{y}_{\alpha}), \alpha(\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}), X)$$

$$= H(\mathbf{x}_{\alpha}, u(\mathbf{x}_{\alpha}), \alpha(\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}), X) - H(\mathbf{y}_{\alpha}, v(\mathbf{y}_{\alpha}), \alpha(\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}), Y)$$

$$+ H(\mathbf{y}_{\alpha}, v(\mathbf{y}_{\alpha}), \alpha(\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}), Y) - H(\mathbf{x}_{\alpha}, v(\mathbf{y}_{\alpha}), \alpha(\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}), X)$$

$$\le \omega \left(\alpha \|\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}\|_{2}^{2} + \|\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}\|_{2}^{2} \right) \quad (19)$$

for all $\alpha \geq 0$, where we used (18) to replace the first two terms to zero in the last inequality. But by taking the $\alpha \to \infty$ limit, $\omega (\alpha \| \mathbf{x}_{\alpha} - \mathbf{y}_{\alpha} \|_{2}^{2} + \| \mathbf{x}_{\alpha} - \mathbf{y}_{\alpha} \|_{2}) \to 0$, while the inequality above specifies that it is bounded from zero by $\min\{1, r\}\delta$, which is a contradiction. In other words, we have $u \leq v$ over the entire \mathbb{R}^{2} . Therefore, if $u : \mathbb{R}^{2} \to \mathbb{R}$ and $v : \mathbb{R}^{2} \to \mathbb{R}$ are both viscosity solution to (15) with the specified boundary conditions: $\lim_{t\to\pm\infty}(u-v) =$ $0, \lim_{x\to\pm\infty}(u-v) = 0$, then u = v over the entire \mathbb{R}^{2} .

Part 2 (the value function is a viscosity solution):

First, we show that V^B is continuous. Suppose that $\mathbf{x}_0 = (t, x_0), \mathbf{x}_1 = (t, x_1) \in \mathbb{R}^2$ are given. For any $\varepsilon > 0$, we can find $\tau_a \in \mathcal{T}$ for a = 0, 1 such that: $V^B(\mathbf{x}_a) \leq \mathcal{V}^B(\mathbf{x}_a; \tau_a, p) + \varepsilon$, while $V^B(\mathbf{x}_a) \geq \mathcal{V}^B(\mathbf{x}_a; \tau_{1-a}, p)$ by definition. It follows that:

$$|V^{B}(\mathbf{x}_{0}) - V^{B}(\mathbf{x}_{1})| \leq \max_{a=0,1} \mathbb{E} \left[e^{-r(\tau_{a}-t)} \left(\max\{v_{\tau_{a}}^{t,x_{a}} - p_{\tau_{a}}, 0\} - \max\{v_{\tau_{a}}^{t,x_{1-a}} - p_{\tau_{a}}, 0\} \right) |\mathscr{F}_{t}] + \varepsilon.$$

Using the Lipschitz condition (2) with Gronwall's inequality to upper bounds the growth of the SDE solutions $\{v_s^{t,x_a}\}_{s\geq t}$, it is possible to show that the RHS approaches zero as $x_1 \to x_0$. Suppose that $\mathbf{x}_0 = (t_0, x), \mathbf{x}_1 = (t_1, x) \in \mathbb{R}^2$ are given, for some $t_1 > t_0$. Then by the optimality principle, we can find $\tau \in \mathcal{T}$ such that

$$\mathbb{E}\left[e^{-r(t_{1}\wedge\tau-t)}V^{B}(t_{1}\wedge\tau,v_{t_{1}\wedge\tau}^{t_{0},x})-V^{B}(\mathbf{x}_{1})-\int_{t_{0}}^{t_{1}\wedge\tau}ce^{-r(s-t_{0})}ds|\mathscr{F}_{t_{0}}\right]+\varepsilon(t_{1}-t_{0})$$

$$\geq V^{B}(\mathbf{x}_{0})-V^{B}(\mathbf{x}_{1})\geq \mathbb{E}\left[e^{-r(t_{1}-t)}V^{B}(t_{1},v_{t_{1}}^{t_{0},x})-V^{B}(\mathbf{x}_{1})-\int_{t_{0}}^{t_{1}}ce^{-r(s-t_{0})}ds|\mathscr{F}_{t_{0}}\right]$$

Using the continuity of V^B in x we have previously proven, we have a pointwise convergence to zero for both integrands as $t_1 \searrow t_0$. We can conclude using the Dominated Convergence Theorem that both expected values converge to zero as $t_1 \searrow t_0$. The convergence to zero as $t_1 \nearrow t_0$ can be obtained similarly. Putting both results together, we find that V^B is continuous in both t and x. Next, we will show that V^B is both a viscosity subsolution and supersolution using the standard argument (e.g. see Yong and Zhou (2012)).

Given $\mathbf{x} = (t, x) \in \mathbb{R}^2$ and a twice continuously differentiable function ϕ such that \mathbf{x} is a local maximum of $V^B - \phi$. Let us assume that $V^B(t, x) > \max\{x - p_t, 0\}$, i.e. \mathbf{x} is in the learning region, otherwise we have $\min_{i=2,3} A_i(t, x, V^B(t, x), \nabla\phi, \Delta\phi) \leq 0$ then it is trivial that $H(t, x, V^B(t, x), \nabla\phi, \Delta\phi) \leq 0$. Let $t' > t \geq 0$, we can find $\tau \in \mathcal{T}$ such that $V^B(t, x) \leq \mathbb{E}\left[e^{-r(t'\wedge\tau-t)}V^B(t'\wedge\tau, v_{t'\wedge\tau}^{t,x}) - \int_t^{t'\wedge\tau} ce^{-r(s-t)}ds|\mathscr{F}_t\right] + \varepsilon(t'-t)$. Then for all t' sufficiently close to t, we have

$$\begin{split} 0 &\leq \mathbb{E}\left[V^B(t,x) - \phi(t,x) - V^B(t',v_{t'}^{t,x}) + \phi(t',v_{t'}^{t,x}) \mid \mathscr{F}_t\right] \leq \mathbb{E}\left[\phi(t' \wedge \tau, v_{t'\wedge\tau}^{t,x}) - \phi(t,x) \right. \\ &\left. - \int_t^{t'\wedge\tau} ce^{-r(s-t)} ds - \left(1 - e^{-r(t'\wedge\tau-t)}\right) V^B(t' \wedge \tau, v_{t'\wedge\tau}^{t,x}) \mid \mathscr{F}_t\right] + \varepsilon(t'-t) \\ &= \mathbb{E}\left[\int_t^{t'\wedge\tau} \left(\partial_t \phi(s, v_s^{t,x}) + \frac{\sigma(v_s)^2}{2} \partial_x^2 \phi(s, v_s^{t,x})\right) ds + \int_t^{t'\wedge\tau} \partial_x \phi(s, v_s^{t,x}) dv_s^{t,x} \right. \\ &\left. - \int_t^{t'\wedge\tau} ce^{-r(s-t)} ds - \left(1 - e^{-r(t'\wedge\tau-t)}\right) V^B(t' \wedge \tau, v_{t'\wedge\tau}^{t,x}) \mid \mathscr{F}_t\right] + \varepsilon(t'-t) \end{split}$$

Since $\{v_t^{t,x}\}_{t\geq 0}$ is a square-integrable martingale, we know that $\int_t^{t'\wedge\tau} \partial_x \phi(s, v_s^{t,x}) dv_s^{t,x}$ is also a continuous square-integrable martingale (see Karatzas and Shreve (2012)⁵), hence the second term vanishes by the Martingale Stopping Theorem. Dividing both-sides by t' - t, taking the limit $t' \to t$ and apply the Dominated Convergence Theorem, we get: $A_1(t, x, V^B(t, x), \nabla \phi, \Delta \phi) \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that $A_1(t, x, V^B(t, x), \nabla \phi, \Delta \phi) \leq 0$. Thus, we have $H(t, x, V^B(t, x), \nabla \phi, \Delta \phi) \leq 0$, so V^B is a viscosity subsolution of (15).

Given $\mathbf{x} = (t, x) \in \mathbb{R}^2$ and a twice continuously differentiable function ϕ such that \mathbf{x} is a local minimum of $V^B - \phi$. Then for all $t' > t \ge 0$ sufficiently close, we have

$$0 \ge \frac{1}{t'-t} \mathbb{E}\left[V^B(t,x) - \phi(t,x) - V^B(t',v_{t'}^{t,x}) + \phi(t',v_{t'}^{t,x}) \mid \mathscr{F}_t \right]$$

⁵ We also need a square–integrability condition: $\mathbb{E}\left[\int_{t}^{t'\wedge\tau} \left(\sigma(v_{s}^{t,x})\partial_{x}\phi(s,v_{s}^{t,x})\right)^{2} ds\right] < \infty$, but we can always ensure this by choosing a more appropriate ϕ with the same $\nabla \phi$ and $\Delta \phi$ at (t,x).

$$\geq \frac{1}{t'-t} \mathbb{E} \left[\phi(t', v_{t'}^{t,x}) - \phi(t,x) - \int_{t}^{t'} c e^{-r(s-t)} ds - \left(1 - e^{-r(t'-t)}\right) V^{B}(t', v_{t'}^{t,x}) \mid \mathscr{F}_{t} \right] \\ = \frac{1}{t'-t} \mathbb{E} \left[\int_{t}^{t'} \left(\partial_{t} \phi(s, v_{s}^{t,x}) + \frac{\sigma(v_{s}^{t,x})^{2}}{2} \partial_{x}^{2} \phi(s, v_{s}^{t,x}) \right) ds + \int_{t}^{t'} \partial_{x} \phi(s, v_{s}^{t,x}) dv_{s}^{t,x} \\ - \int_{t}^{t'} c e^{-r(s-t)} ds - \left(1 - e^{-r(t'-t)}\right) V^{B}(t', v_{t'}^{t,x}) \mid \mathscr{F}_{t} \right].$$

The second inequality followed from the optimality principle: $V^B(t,x) \geq \mathbb{E}\left[e^{-r(t'-t)}V^B(t',v_{t'}^{t,x}) - \int_t^{t'} ce^{-r(s-t)}ds \mid \mathscr{F}_t\right]$. The second term is a continuous squareintegrable martingale, hence vanishes as explained. Taking limit $t' \to t$ and apply the Dominated Convergence Theorem, we get: $A_1(t,x,V^B(t,x),\nabla\phi,\Delta\phi) \geq 0$. The conditions $A_{i=2,3}(t,x,V^B(t,x),\nabla\phi,\Delta\phi) \geq 0$ only depends on $V^B(t,x) \geq \max\{x-p_t,0\}$, hence they are trivial. Thus, we have shown that $H(t,x,V^B(t,x),\nabla\phi,\Delta\phi) \geq 0$, so V^B is a viscosity supersolution of (15).

Working directly with the viscosity solution via definition 2 can still be challenging, thus we may alternatively consider the following free-boundary backward parabolic PDE boundary value problem: Find $V : [0, T] \times \mathbb{R} \to \mathbb{R}$, and continuously differentiable functions $\bar{V}[p], \underline{V}[p] : \mathbb{R} \to \mathbb{R}$ satisfying $\bar{V}_t[p] \geq \underline{V}_t[p]$, such that

$$\begin{cases} \frac{\sigma(x)^2}{2} \partial_x^2 V(t,x) + \partial_t V(t,x) - rV(t,x) - c = 0, & (t,x) \in \Omega \\ V(t,\bar{V}_t[p]) = \bar{V}_t[p] - p_t, & \partial_x V(t,\bar{V}_t[p]) = 1, \\ V(t,\underline{V}_t[p]) = 0, & \partial_x V(t,\underline{V}_t[p]) = 0, \\ V(T,x) = V_0^B(x;p_T), \end{cases}$$
(20)

where

$$\Omega := \{ (t, x) \in (-\infty, T] \times \mathbb{R} \mid \underline{V}_t[p] < x < \overline{V}_t[p] \}.$$

This PDE connects us back to the constant price benchmark calculations in §3.1, although now we have the moving purchase and exit boundaries $\overline{V}[p]$ and $\underline{V}[p]$ instead of the fixed counterpart back in §3.1. The second and the third lines of (20) amount to the value-matching and the smooth-pasting conditions at the purchase and exit boundaries, respectively. Given a solution V to (20) on Ω with the specified boundary conditions, we can extend it to \widetilde{V} , a function continuously differentiable on \mathbb{R}^2 , and twice continuously differentiable in x on $\mathbb{R}^2 \setminus \partial \Omega$, by defining $\widetilde{V}(t,x) = \max\{x-p_t, 0\}$ if $t \leq T$ and $x \notin (\underline{V}_t[p], \overline{V}_t[p])$, and $\widetilde{V}(t,x) = V_0^B(x; p_T)$ if t > T. This extension is rather natural, therefore, we will abuse the notation and simply refer to \widetilde{V} as V. The upshot is that the solution V will coincides with the consumer's value function V^B , as we state formally below in Lemma 3. This justifies that the constant price benchmark solutions we solved in §3.1 are the viscosity solutions, hence the value functions of their respective consumer's problems.

Solving (20) in full generality is beyond the scope of this research. For an arbitrary given pricing policy $p \in \mathcal{P}_T$, it is likely that there exists no analytical closed-form viscosity solution. However, if p is a small perturbation from a nice policy with a known solution, then we expect the solution corresponding to p to be a small perturbation from the known solution. The PDE formulation of the problem (20) enables us to employ the perturbation theory. Suppose that we know value function $V^B(.,.;p)$ for a given $p \in \mathcal{P}_T$ is a solution to (20), and we would like to compute $V^B(.,.;p+\sqrt{\varepsilon}h)$ for some $h \in \mathcal{P}_T$ and a small $\varepsilon > 0$. By Lemma 3, we aim to solve for the corresponding PDE solution $V(., .; p+\sqrt{\varepsilon}h)$. The idea of perturbation theory is to proceed by writing $V(.,.; p + \sqrt{\varepsilon}h) = V_0(.,.) + V_1(.,.)\sqrt{\varepsilon} + V_2(.,.)\varepsilon + \cdots$, where $V_0(.,.) := V^B(.,.;p)$, and $\bar{V}_t[p+\sqrt{\varepsilon}h] = \bar{V}_{0,t} + \bar{V}_{1,t}\sqrt{\varepsilon} + \bar{V}_{2,t}\varepsilon + \cdots, \underline{V}_t[p+\sqrt{\varepsilon}h] = \underline{V}_{0,t} + \underline{V}_{1,t}\sqrt{\varepsilon} + \underline{V}_{2,t}\varepsilon + \cdots,$ where $\bar{V}_{0,t} := \bar{V}_t[p], \underline{V}_{0,t} := \underline{V}_t[p]$. By substituting these expansions into (20) and comparing the $\varepsilon^{k/2}$ terms for $k = 1, 2, \cdots$, we can solve for $V_k, \overline{V}_k, \underline{V}_k$ using the knowledge of $V_{k'}, \overline{V}_{k'}, \underline{V}_{k'}$ for $k' = 0, \dots, k-1$. In general, the validity of this procedure relies on the convergence with some positive radius of all the $\sqrt{\varepsilon}$ -power series involved. The readers are welcome to accept this as an assumption and skip the technical detail in the next paragraph and the second part of Lemma 3. However, we also provide a self-contained comparison principle argument to formalize the process compatible with the ε -equilibrium concept, as we will state below in Lemma 3.

Suppose that $V_{\leq k}^{\varepsilon}(.,.;p+\sqrt{\epsilon}h) := V_0(.,.) + V_1(.,.)\sqrt{\varepsilon} + \cdots + V_k(.,.)\varepsilon^{k/2}, \ \overline{V}_{\leq k,t}^{\varepsilon}[p+\sqrt{\varepsilon}h] := \overline{V}_{0,t} + \overline{V}_{1,t}\sqrt{\varepsilon} + \cdots + \overline{V}_{k,t}\varepsilon^{k/2}, \ \text{and} \ \underline{V}_{\leq k,t}^{\varepsilon}[p+\sqrt{\varepsilon}h] := \underline{V}_{0,t} + \underline{V}_{1,t}\sqrt{\varepsilon} + \cdots + \underline{V}_{k,t}\varepsilon^{k/2}, \ \text{satisfies}$ (20) over $\Omega_{\leq k}^{\varepsilon} := \{(t,x) \in (-\infty,T] \times \mathbb{R} | \underline{V}_{\leq k,t}^{\varepsilon} < x < \overline{V}_{t,\leq k}^{\varepsilon} \}$ up to the $\varepsilon^{(k+1)/2}$ -order. Note that although both the value-matching and smooth-pasting conditions are only satisfied to the $\varepsilon^{(k+1)/2}$ -order, it is possible to find a twice continuously differentiable function $\chi : \mathbb{R}^2 \setminus \Omega_{\leq k}^{\varepsilon} \to \mathbb{R}$ which continuously differentiably transitions from $V_{\leq k}^{\varepsilon}$ at $\partial \Omega_{\leq k}^{\varepsilon}$ to $\max\{x - p_t, 0\}$ for all (t,x) some distance R > 0 away from $\Omega_{\leq k}^{\varepsilon}$, e.g. a smooth 'bump' function. In particular, we have $\chi = V_{\leq k}^{\varepsilon}$ and $\nabla \chi = \nabla V_{\leq k}^{\varepsilon}$ on $\partial \Omega_{\leq k}^{\varepsilon}$, and additionally we require that $|\partial_t \chi(t,x) - p'_t - \sqrt{\varepsilon}h'_t| = O(\varepsilon^{(k+1)/2}), \ |\partial_x^2 \chi| = O(\varepsilon^{(k+1)/2}), \ and that the asymptotic boundary conditions (16) are met. We extend <math>V_{\leq k}^{\varepsilon}$ to $\widetilde{V}_{\leq k}^{\varepsilon}$, a function continuously differentiable in x on $\mathbb{R}^2 \setminus \partial \Omega_{\leq k}^{\varepsilon}$, by defining $\widetilde{V}_{\leq k}^{\varepsilon}(t,x) = \chi(t,x)$ if $t \leq T$ and $x \notin (\underline{V}_{\leq k,t}^{\varepsilon}, \overline{V}_{\leq k,t}^{\varepsilon}), \ and \widetilde{V}_{\leq k}^{\varepsilon}(t,x) = V_0^B(x; p_T + \sqrt{\varepsilon}h_T)$ if t > T. We shall abuse the notation and simply refer to $\widetilde{V}_{\leq k}^{\varepsilon}$ as $V_{\leq k}^{\varepsilon}$.

Lemma 3. Consider $p, h \in \mathcal{P}_T$ pricing strategies and a given $\varepsilon > 0$.

- 1. If V satisfies the free-boundary backward parabolic PDE boundary value problem (20) with the pricing policy $p \in \mathcal{P}_T$, such that $V(t, x) \ge \max\{x - p_t, 0\}$, and $p'_t + r(\bar{V}_t[p] - p_t) + c > 0$ for all $(t, x) \in \mathbb{R}^2$, then V is a viscosity solution to (15). In particular, the consumer's value function is given by $V^B = V$.
- 2. If $V_{\leq k}^{\varepsilon}$ satisfies the free-boundary backward parabolic PDE boundary value problem (20) up to the $\varepsilon^{(k+1)/2}$ -order with the pricing policy $p + \sqrt{\varepsilon}h \in \mathcal{P}_T$, such that $V_{\leq k}^{\varepsilon}(t,x) \geq \max\{x - p_t, 0\} + O(\varepsilon^{(k+1)/2})$, and $p'_t + \sqrt{\varepsilon}h'_t + r(\bar{V}_t[p + \sqrt{\varepsilon}h] - p_t) + c > O(\varepsilon^{(k+1)/2})$ for all $(t, x) \in \mathbb{R}^2$, then $V^B = V_{\leq k}^{\varepsilon} + O(\varepsilon^{(k+1)/2})$.

Proof. Part 1:

Let such a solution V to (20) be given. Since we have assumed $V(t, x) \ge \max\{x - p_t, 0\}$ and $p'_t + r(\bar{V}_t[p] - p_t) + c > 0$, then $V - \max\{x - p_t, 0\} \ge 0$ for all $\mathbf{x} \in \Omega$, and $c + rV - \partial_t V - \frac{\sigma(\mathbf{x})^2}{2}\partial_x^2 V \ge 0$ for all $\mathbf{x} \in \mathbb{R}^2 \setminus \Omega$. By the value–matching, the smooth pasting conditions, and the assumption that $p \in \mathcal{P}_T$ is smooth, we have that V is continuously differentiable⁶. Moreover, V is twice continuously differentiable in x on $\mathbb{R}^2 \setminus \partial\Omega$, as it is a (classical) solution to the PDE on Ω , and $\max\{x - p_t, 0\}$ is twice continuously differentiable in x on $\mathbb{R}^2 \setminus \Omega$. Therefore, we have $H(\mathbf{x}, V, \nabla V, \Delta V) = 0$ classically on $\mathbb{R}^2 \setminus \partial\Omega$. Thus, for any twice continuously differentiable ϕ and any $\mathbf{x}_0 \in \mathbb{R}^2$, we have $\nabla \phi(\mathbf{x}_0) = \nabla V(\mathbf{x}_0)$, and we can find $\{\mathbf{x}_i\}_{i=0}^{\infty} \subset \mathbb{R}^2 \setminus \partial\Omega$ converging to \mathbf{x}_0 . If \mathbf{x}_0 is a local maximum of $V - \phi$ then $\partial_x^2 \phi(\mathbf{x}_0) \ge \lim_{i \to \infty} \partial_x^2 V(\mathbf{x}_i)$ which implies $H(\mathbf{x}_0, V(\mathbf{x}_0), \nabla \phi(\mathbf{x}_0), \Delta \phi(\mathbf{x}_0)) \le \lim_{i \to 0} H(\mathbf{x}_i, V(\mathbf{x}_i), \nabla V(\mathbf{x}_i), \Delta V(\mathbf{x}_i)) = 0$. Similarly, if \mathbf{x}_0 is a local minimum of $V - \phi$ then $\partial_x^2 \phi(\mathbf{x}_0) \le \lim_{i \to \infty} \partial_x^2 V(\mathbf{x}_i)$ which implies $H(\mathbf{x}_0, V(\mathbf{x}_0), \nabla \phi(\mathbf{x}_0), \Delta \phi(\mathbf{x}_0)) \ge \lim_{i \to \infty} H(\mathbf{x}_i, V(\mathbf{x}_i), \nabla V(\mathbf{x}_i), \Delta V(\mathbf{x}_i)) = 0$.

Part 2:

Repeat the argument from the previous part with the perturbed pricing policy $p + \sqrt{\varepsilon}h \in \mathcal{P}_T$, we have that $H(\mathbf{x}, V_{\leq k}^{\varepsilon}, \nabla V_{\leq k}^{\varepsilon}, \Delta V_{\leq k}^{\varepsilon}) = O(\varepsilon^{(k+1)/2})$ classically on $\mathbb{R}^2 \setminus \partial \Omega_{\leq k}^{\varepsilon}$. Moreover, for any twice continuously differentiable ϕ and any $\mathbf{x}_0 \in \mathbb{R}^2$, if \mathbf{x}_0 is a local maximum of $V_{\leq k}^{\varepsilon} - \phi$ then $H(\mathbf{x}_0, V_{\leq k}^{\varepsilon}, \nabla\phi, \Delta\phi) \leq O(\varepsilon^{(k+1)/2})$, and if \mathbf{x}_0 is a local minimum of $V_{\leq k}^{\varepsilon} - \phi$ then $H(\mathbf{x}_0, V_{\leq k}^{\varepsilon}, \nabla\phi, \Delta\phi) \geq O(\varepsilon^{(k+1)/2})$. Since $V_{\leq k}^{\varepsilon}$ satisfies the same asymptotic boundary conditions as the value function V^B , we can repeat the comparison principle argument in the proof of Lemma 2. In particular, setting $u := V_{\leq k}^{\varepsilon}, v := V^B$ we have (18) becomes $H(\mathbf{x}_{\alpha}, V_{\leq k}^{\varepsilon}, \alpha(\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}), X) + O(\varepsilon^{(k+1)/2}) \leq 0 \leq H(\mathbf{y}_{\alpha}, V^B, \alpha(\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}), Y)$, which means (19) becomes $\min\{1, r\}(V_{\leq k}^{\varepsilon}(\mathbf{x}_{\alpha}) - V^B(\mathbf{y}_{\alpha})) \leq \omega(\alpha \|\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}\|_2^2 + \|\mathbf{x}_{\alpha} - \mathbf{y}_{\alpha}\|_2) + O(\varepsilon^{(k+1)/2})$. Taking the limit $\alpha \to \infty$, we find that $\sup_{\mathbf{x} \in \mathbb{R}^2} \left(V_{\leq k}^{\varepsilon}(\mathbf{x}) - V^B(\mathbf{x})\right) \leq O(\varepsilon^{(k+1)/2})$, in other

⁶ To get the continuity of t derivative across the boundary, consider the defining equation: $V^B(t, \bar{V}_t[p]; p) = \bar{V}_t[p] - p_t$. Differentiating with respect to t gives: $\bar{V}_t'[p] \cdot \partial_x V^B(t, \bar{V}_t[p]; p) + \partial_t V^B(t, \bar{V}_t[p]; p) = \bar{V}_t'[p] - p_t'$, or $\partial_t V^B(t, \bar{V}_t[p]; p) = -p_t'$. Similarly, we have $\partial_t V^B(t, \underline{V}_t[p]; p) = 0$

words: $V_{\leq k}^{\varepsilon}(\mathbf{x}) \leq V^B(\mathbf{x}) + O(\varepsilon^{(k+1)/2})$. On the other hand, setting $u := V^B, v := V_{\leq k}^{\varepsilon}$ yields $V^B(\mathbf{x}) \leq V_{\leq k}^{\varepsilon}(\mathbf{x}) + O(\varepsilon^{(k+1)/2})$, thus we have $V^B(\mathbf{x}) = V_{\leq k}^{\varepsilon}(\mathbf{x}) + O(\varepsilon^{(k+1)/2})$ as claimed. \Box

The conditions on p' and h' in Lemma 3 are sufficient but not necessary. They should not be too restrictive for us as we will mainly consider small perturbations from the known constant price solution and investigate the direction of consumers' reactions. We will use k = 1 in our application of perturbation theory, solving (20) up to the $\sqrt{\varepsilon}$ -order, to be consistent with the ε -equilibrium concept. In other words, we have

$$V^{B}(t,x;p+\sqrt{\varepsilon}h) = V^{B}(t,x;p) + V_{1}(t,x)\sqrt{\varepsilon} + O(\varepsilon),$$

and we can take the consumer's learning policy to be given by the corresponding boundaries $\bar{V}[p + \sqrt{\varepsilon}h] = \bar{V}[p] + \bar{V}_1\sqrt{\varepsilon} + O(\varepsilon)$, and $\underline{V}[p + \sqrt{\varepsilon}h] = \underline{V}[p] + \underline{V}_1\sqrt{\varepsilon} + O(\varepsilon)$.

Lemma 3 also shows an $\sqrt{\varepsilon}$ -order changes in p will results in $\sqrt{\varepsilon}$ -order changes in the value of V, and the boundaries $\overline{V}[p], \underline{V}[p]$. The following gives a more concrete upper-bounds:

Lemma 4. Let $p, q \in \mathcal{P}_T$ then $|V^B(t, x; p) - V^B(t, x; q)| \leq \max_{s \in [t,T]} e^{-r(s-t)} |p_s - q_s|$ for all $(t, x) \in \mathbb{R}^2$.

Remark 3. Lemma 4 shows that for discounting consumers r > 0, any changes in price far in the future do not have much effect in the present. This enables us to extend our consumer response results to arbitrary $p \in \mathcal{P} \subset C^{\infty}[0,\infty)$ such that $\lim_{t\to\infty} e^{-rt}p_t = 0$. We find the viscosity solution $V(.,.;p^T)$ of (15) corresponding to $p^T \in \mathcal{P}_T$, which coincides with the value function $V^B(.,.;p^T)$ according to Lemma 2 where p^T is given by p over $[0, T - \varepsilon]$ and constant for all $t \geq T$. Then for all sufficiently large T we have $|V(t,x;p^{T'}) - V(t,x;p^{T''})| < \varepsilon$ for all T', T'' > T. This is a real-valued Cauchy sequence, so we may recover the value function of an infinite horizon p from the limit of the viscosity solution sequence $\{V(.,.;p^T)\}_{T\geq 0}$ which converges uniformly over any compact subset of \mathbb{R}^2 .

The following gives a somewhat general characterization of the $\sqrt{\varepsilon}$ -order perturbed boundaries in terms of the zero-th order solution.

Proposition 2. Consider a given $p \in \mathcal{P}_T$ pricing strategy. Suppose that the consumer's value function $V^B(.,.;p)$ is a solution to the PDE (20) which is smooth on Ω with smooth corresponding purchase and exit boundaries $\overline{V}[p], \underline{V}[p] : \mathbb{R} \to \mathbb{R}$. Let $h \in \mathcal{P}_T$ be arbitrary, then under the pricing strategy $\tilde{p} := p + \sqrt{\varepsilon}h$, we can find an ε -optimal value function taking the form:

$$V^{B}(t,x;\tilde{p}) = V^{B}(t,x-\sqrt{\varepsilon}h_{t};p) + \sqrt{\varepsilon}V^{B}_{1}(t,x) + O(\varepsilon), \qquad (21)$$

where $V_1^B(.,.): \Omega := \{(t,x) \in (-\infty,T] \times \mathbb{R} \mid \underline{V}_t[p] < x < \overline{V}_t[p]\} \to \mathbb{R}$ is given by:

$$V_1^B(t,x) = -\mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} h'_s e^{-r(s-t)} \partial_x V^B(s, v_s^{t,x}; p) ds |\mathscr{F}_t\right] \\ + \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} h_s e^{-r(s-t)} \sigma(v_s^{t,x}) \sigma'(v_s^{t,x}) \partial_x^2 V^B(s, v_s^{t,x}; p) ds |\mathscr{F}_t\right], \quad (22)$$

and $\tau_{\Omega}^{t,x} := \inf\{t' \ge t \mid (t', v_{t'}^{t,x}) \notin \Omega\} \le T$ is the exit time. We can find the ε -optimal purchase and exit boundaries taking the form:

$$\bar{V}[\tilde{p}] = (\bar{V}[p] + \sqrt{\varepsilon}h) + \sqrt{\varepsilon}\bar{R} + O(\varepsilon)
\underline{V}[\tilde{p}] = (\underline{V}[p] + \sqrt{\varepsilon}h) + \sqrt{\varepsilon}\underline{R} + O(\varepsilon)$$
(23)

for functions $\overline{R} : \mathbb{R} \to \mathbb{R}$, and $\underline{R} : \mathbb{R} \to \mathbb{R}$ given by:

$$\bar{R}_t = -\frac{\partial_x V_1^B(t, \bar{V}_t[p])}{\partial_x^2 V^B(t, \bar{V}_t[p]; p)}, \qquad \underline{R}_t = -\frac{\partial_x V_1^B(t, \underline{V}_t[p])}{\partial_x^2 V^B(t, \underline{V}_t[p]; p)}$$

In particular, if $\sigma'(.) = O(\varepsilon)$ (stable volatility), and $h := K\tilde{h}$ for some monotonically increasing $\tilde{h} \in \mathcal{P}_T$ and a constant $K \in \mathbb{R} \setminus \{0\}$, then $\bar{S}_t := \bar{R}_t/K \leq 0$ and $\underline{S}_t := \underline{R}_t/K \geq 0$ for all $t \in \mathbb{R}$.

Remark 4. The smoothness assumptions in Proposition 2 were imposed for simplicity, they are not necessary conditions. As long as the classical solution to the $\sqrt{\varepsilon}$ -order PDE boundary-value problem $V_1(t, x) \in C^{1,2}(\Omega)$ exists, then the results hold.

In our work, we pay special attention to the linear pricing policies. As it turns out, when p is linear in t, the consumer's value function admits an analytic closed-form under some simple learning settings, such as when $\{v_t\}_{t\geq 0}$ is a vanilla Brownian motion. It is also simpler to analyze the firm's strategies restricted to the space of linear pricing. The fact that the space of linear pricing is much smaller than the general pricing space also simplifies the problem, especially when searching for the firm's optimal pricing strategy later in §4. Consideration of linear pricing may seem restrictive, but the following result, which is a simple application of Lemma 4, shows that for myopic enough ε -optimal consumers, any pricing strategies which is sufficiently slow-moving can be approximated by linear pricing. Intuitively, unless the price changes very drastically in the far future such as growing super-exponentially, the myopic consumers do not look too far into the future, and over any sufficiently short time interval any differentiable functions *look like* a linear function.

Proposition 3. (Almost optimality of linear price approximation) Let $p \in \mathcal{P}$ be an admissible pricing policy with $\sup_{t \in \mathbb{R}} |p_t''| \leq M$. At any $\mathbf{x} = (t, x) \in \mathbb{R}^2$ we consider the linear approximation pricing policy $l_{\mathbf{x}} \in \mathcal{P} : s \mapsto l_{\mathbf{x},s} := p_t + p_t' \cdot (s - t)$. Let the consumer's optimal learning strategy given the linear pricing $l_{\mathbf{x}}$ be $\tau^*[l_{\mathbf{x}}] \in \mathcal{T}$. If the consumer is sufficiently myopic: $r > e^{-1}\sqrt{2M/\varepsilon}$, then $\tau^*[l_{\mathbf{x}}]$ is also the consumer's ε -optimal stopping time under the p pricing strategy:

$$\mathcal{V}^B(t, x; \tau^*[l_{\mathbf{x}}]; p) \ge V^B(t, x; p) - \varepsilon.$$

Remark 5. It is also possible to apply Lemma 4 to the constant price approximation, i.e. we assume $\sup_{t \in \mathbb{R}} |p'_t| \leq M$ and consider $p_0 \in \mathcal{P} : t \mapsto p_0$ for all $t \in \mathbb{R}$. We have $\mathcal{V}^B(t, x; \tau^*[p_0]; p) \geq V^B(t, x; p) - \varepsilon$ if $r > e^{-1}M/\varepsilon$. In other words, if the consumer is very myopic, $r = O(1/\varepsilon)$, then every pricing $p \in \mathcal{P}$ can be treated as constant, which is a trivial result.

We summarize the two simplifying assumptions for reducing any pricing strategies to a linear one. They are by no means minimal. However, they do not substantially limit our contribution.

Simplifying Assumptions and Discussion

Assumption 1. For a given $\varepsilon > 0$, we assume that:

- Consumers are ε -optimal, and is sufficiently myopic
- The firm adjusts the price slowly over time: $|p'_t| \in O(\sqrt{\varepsilon})$

such that the conditions for Proposition 3 are satisfied.

Assumption 1 ensures the validity of our perturbation technique and the consistency with the ε -equilibrium concept. It tells us when it is justified for us to apply a linear pricing result in our work to more general non-linear pricing strategies. For a given price function $p \in \mathcal{P}_T$ such that Assumption 1 is satisfied (i.e. r >> 0 is large, and $|p'_t|$ is small) the consumer will derive the learning strategy from the linear pricing approximation:

$$t \mapsto p_0 + \sqrt{\varepsilon}Kt, \qquad \sqrt{\varepsilon}K := p'_0 \in O(\sqrt{\varepsilon}).$$
 (24)

based on Proposition 3. Of course, such linear pricing does not belong to \mathcal{P}_T for any T > 0, however, this is not a problem according to Remark 3. With assumption 1, the consumer remains sufficiently forward-looking and rationally anticipates the price evolution for us to capture the equilibrium response to the firm's non-stationary pricing.

Solution: Learning product attributes

We revisit the learning process in §3.1.1 but with the linear pricing (24). This setup is a rare example where the free-boundary PDE (20) can be solved exactly, thus giving the exact value function according to the first part of Lemma 3. The reason lies in the fact that $\sigma(.)$ is a constant in this case, thus, the probability measure of $\{v_s^{t,x}\}_{s\geq t}$ is *x*-translation invariant. Therefore, we can transform to the simpler frame of reference where the price is fixed at p_0 while the consumer valuation process is a drifted Brownian motion $v_t = -\sqrt{\varepsilon}Kt + \sigma W_t$. The transformed problem is stationary in time, with the corresponding HJB

$$\frac{\sigma^2}{2}\partial_x^2 V(x) - \sqrt{\varepsilon}K\partial_x V(x) - rV(x) - c = 0.$$

Therefore, the free–boundary problem (20) can be solved in this case by first solving the HJB above, before making an inverse transformation back to the original frame of reference.

Proposition 4. Consider the learning process (8). Under a linear pricing strategy $p: t \mapsto p_0 + \sqrt{\varepsilon}Kt \in \mathcal{P}$, and such that Assumption 1 is satisfied, the consumer's value function is given by

$$V^{B}(t,x) = A_{1}e^{\frac{\sqrt{\varepsilon}K - \sqrt{\varepsilon}K^{2} + 2r\sigma^{2}}{\sigma^{2}}(x - p_{0} - \sqrt{\varepsilon}Kt)} + A_{2}e^{\frac{\sqrt{\varepsilon}K + \sqrt{\varepsilon}K^{2} + 2r\sigma^{2}}{\sigma^{2}}(x - p_{0} - \sqrt{\varepsilon}Kt)} - \frac{c}{r}$$
(25)

with purchase and exit boundaries given by

$$\bar{V}_t = p_0 + \bar{V}[\sqrt{\varepsilon}K] + \sqrt{\varepsilon}Kt, \qquad \underline{V}_t = p_0 + \underline{V}[\sqrt{\varepsilon}K] + \sqrt{\varepsilon}Kt$$
(26)

where the constants $\overline{V}[\sqrt{\varepsilon}K]$, $\underline{V}[\sqrt{\varepsilon}K]$, A_1 , and A_2 are determined by boundary conditions in the appendix. To the $\sqrt{\varepsilon}$ -order, $\overline{V}[\sqrt{\varepsilon}K]$ and $\underline{V}[\sqrt{\varepsilon}K]$ take the following analytical form,

$$\bar{V}[\sqrt{\varepsilon}K] = \bar{V} + \sqrt{\varepsilon}\bar{R} + O(\varepsilon), \qquad \underline{V}[\sqrt{\varepsilon}K] = \underline{V} + \sqrt{\varepsilon}\underline{R} + O(\varepsilon), \tag{27}$$

where

$$\underline{S} := \frac{R}{K} = \left(\frac{\bar{V} - V}{\sigma^2}\right) \left(\bar{V} + \frac{c}{r}\right) - \frac{1}{2r} = \frac{\sqrt{\frac{c^2}{r^2} + \frac{\sigma^2}{2r}}}{\sigma\sqrt{2r}} \log\left(\sqrt{\frac{r\sigma^2}{2c^2}} + \sqrt{1 + \frac{r\sigma^2}{2c^2}}\right) - \frac{1}{2r} > 0$$
$$\bar{S} := \frac{\bar{R}}{K} = \underline{S} - \frac{1}{2r} \cdot \frac{\bar{V} - V}{\bar{V} + c/r} = \frac{1/(\sigma\sqrt{2r})}{\sqrt{\frac{c^2}{r^2} + \frac{\sigma^2}{2r}}} \cdot \frac{c^2}{r^2} \log\left(\sqrt{\frac{r\sigma^2}{2c^2}} + \sqrt{1 + \frac{r\sigma^2}{2c^2}}\right) - \frac{1}{2r} < 0.$$

and $\overline{V}, \underline{V}$ are given by (10), (11), respectively.

Compared to the result of Proposition 2, we have that R, \underline{R} are constant in this case. Compared to the constant price benchmark, an increasing pricing scheme (K > 0) with the same initial price has two impacts on the purchasing threshold. On the one hand, the benefit of learning becomes lower because the consumer needs to pay more in the future if she receives positive information and likes the product more. Rationally anticipating this, the consumer has a lower incentive to search and is more inclined to purchase now, reducing the purchasing threshold (captured by the negative $\sqrt{\varepsilon}K\bar{S}$ term). On the other hand, a higher price makes the consumer less willing to purchase, raising the purchasing threshold (captured by the positive $\sqrt{\varepsilon}Kt$ term). Since the first effect remains stable while the second effect increases over time, the purchasing threshold is lower than the benchmark threshold at the beginning but eventually exceeds the benchmark threshold as the price keeps increasing.



Figure 2: Purchasing and quitting boundaries when $c = .2, p = 1, r = .1, \sigma = 1, \epsilon = 0.01$, and K = 1.

An increasing pricing scheme also has two impacts on the quitting threshold. Both a lower benefit of searching and a higher price make it more likely for the consumer to quit. So, the quitting threshold is always higher than the benchmark threshold. We also find that the consumer searches in a narrower region (smaller $\bar{V}_t - \underline{V}_t$) if the price increases rather than staying constant because of the lower benefit of searching. Figure 2 illustrates the purchasing and quitting boundaries in this case, under both non-stationary pricing and constant price.

A decreasing pricing scheme (K < 0) has the opposite impact on the purchasing and quitting thresholds. The purchasing threshold is higher than the benchmark threshold at the beginning because the consumer has a stronger incentive to search and is less inclined to purchase immediately. It eventually falls below the benchmark threshold as the price keeps



Figure 3: Purchasing and quitting boundaries when $c = .2, p = 1, r = .1, \sigma = 1, \epsilon = 0.01$, and K = -1.

decreasing. The quitting threshold is always lower than the benchmark threshold because the benefit of both searching and purchasing is higher. Also, the consumer searches in a broader region. Figure 3 illustrates the purchasing and quitting boundaries in this case, under both non-stationary pricing and constant price.

Solution: Binary classification

We revisit the learning process in §3.1.2 but with the linear pricing (24). This example cannot be solved exactly, therefore we consider the perturbation ansatz: $V^B(t, x; p_0 + \sqrt{\varepsilon}Kt) = V_0^B(x - \sqrt{\varepsilon}Kt; p_0) + \sqrt{\varepsilon}V_1^B(x) + O(\varepsilon)$. Substituting this into (20), and collecting the $\sqrt{\varepsilon}$ terms, we obtain the needed equations for V_1^B , and the $\sqrt{\varepsilon}$ -order boundaries \bar{V}_1 , \underline{V}_1 . As we shall see, our choice of ansatz means the problem of solving for V_1^B , \bar{V}_1 , \underline{V}_1 reduces to a backward parabolic PDE initial boundary value problem, with the fixed boundaries given by $\bar{V}[p_0], \underline{V}[p_0]$.

Proposition 5. Consider the learning process (12). Under a linear pricing strategy $p: t \mapsto p_0 + \sqrt{\varepsilon}Kt \in \mathcal{P}$, and such that Assumption 1 is satisfied, there is an ε -optimal consumer learning strategy with the value function and the boundaries taking the form:

$$V^{B}(t,x) = V_{0}^{B}(x - \sqrt{\varepsilon}Kt; p_{0}) + \sqrt{\varepsilon}V_{1}^{B}(x) + O(\varepsilon)$$
$$\bar{V}_{1,t} = p_{0} + \bar{V} + \sqrt{\varepsilon}(Kt + \bar{R}_{t})$$
$$\underline{V}_{1,t} = p_{0} + \underline{V} + \sqrt{\varepsilon}(Kt + \underline{R}_{t})$$

where $V_0^B(.; p_0)$ is given by (13); $V_1^B(.)$ is given by:

$$\begin{split} V_1^B(t,x) &= -K \cdot \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} e^{-r(s-t)} \partial_x V^B(s,v_s^{t,x};p_0) ds |\mathscr{F}_t\right] \\ &\quad + \frac{K}{\sigma^2} \cdot \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} s e^{-r(s-t)} v_s^{t,x} (1-v_s^{t,x}) (1-2v_s^{t,x}) \partial_x^2 V^B(s,v_s^{t,x};p_0) ds |\mathscr{F}_t\right], \end{split}$$

and $\bar{R}_t := -\frac{\partial_x V_1^B(t, p_0 + \bar{V})}{\partial_x^2 V^B(t, p_0 + \bar{V}; p_0)}, \ \underline{R}_t := -\frac{\partial_x V_1^B(t, p_0 + \underline{V})}{\partial_x^2 V^B(t, p_0 + \underline{V}; p_0)}.$

4 Firm's Strategy

4.1 Firm's Expected Payoff

The expected payoff for the firm implementing the pricing strategy $p \in \mathcal{P}$ with marginal cost g is given by $\mathcal{V}^S(x; \tau^*[p], p)$ as given by (4) where $\tau^*[p] \in \mathcal{T}$ denotes the consumer's ε -optimal response to p. The formula (4) is rather abstract, in this section we show how to compute $\mathcal{V}^S(x; \tau^*[p], p)$ which we shall denote by $\mathcal{V}^S(x; p)$ hereafter for simplicity. For the consumer with initial valuation x, let $\overline{V}[p], \underline{V}[p] : [0, \infty) \to \mathbb{R}$ denotes the consumer's decision boundaries corresponding to the $\tau^*[p]$ learning strategy, we solve the Kolmogorov forward equation with absorbing boundary condition:

$$\begin{cases} \frac{1}{2}\partial_{v}^{2} \left[\sigma(v)^{2} U(t,v;x)\right] - \partial_{t} U(t,v;x) = 0, & (t,v) \in \Omega\\ U(t,\bar{V}_{t}[p];x) = 0, & U(t,\underline{V}_{t}[p];x) = 0 \\ U(t=0,v;x) = \delta(v-x) \end{cases}$$
(28)

Where $\Omega := \{(t, v) \in [0, \infty) \times \mathbb{R} \mid \underline{V}_t[p] < v < \overline{V}_t[p]\}$, and $\delta(v - x)$ denotes the Dirac-Delta distribution concentrated at x. The parabolic initial boundary value problem (28) governs the transition probability U(t, v; x) of a particle starting from x to some point v at later time t as described by the process $\{v_t^x\}_{t\geq 0}$. When it is clear from the context, we may denote U(t, v; x) simply as U(t, v). The existence and properties of the solution U(t, v; x) depends on the smoothness conditions of $\overline{V}[p]$, $\underline{V}[p]$ (see (Friedman, 2008, Chapter 3)). We shall assume that all necessary conditions are satisfied so that the solution $U(t, v; x) \in C^{1,2}(\Omega)$ exists. The probability flux of consumer hitting the moving purchase boundary, thus getting absorbed, at time s is given by

$$-\frac{1}{2}\partial_v \left[\sigma(\bar{V}_t)^2 U(t,\bar{V}_t)\right] - \bar{V}_t' \cdot U(t,\bar{V}_t) = -\frac{1}{2}\partial_v \left[\sigma(\bar{V}_t)^2 U(t,\bar{V}_t)\right].$$

Where the boundary time derivative \bar{V}'_t term is needed to take the boundary movement into account, but the term nevertheless vanishes by the boundary condition: $U(t, \bar{V}_t) = 0$. Hence, if $x \in [\underline{V}_t, \bar{V}_t]$ then we have that

$$\mathcal{V}^S(x;p) = -\frac{1}{2} \int_0^\infty e^{-mt} (p_t - g) \partial_v \left[\sigma(\bar{V}_t)^2 U(t,\bar{V}_t) \right] dt, \tag{29}$$

otherwise, we simply have $\mathcal{V}^S(x;p) = (p_0 - g) \mathbf{1}_{x \ge \overline{V}_0}$, i.e. the consumer purchases immediately and the game ends at t = 0.

4.2 Direction of Price Evolution

The most important property of the firm's optimal pricing strategy is the direction of price evolution. Whether the price should stay constant, increase, or decrease over time? In this section, we carry–on the previous Assumption 1. Unless mentioned otherwise, we will restrict the firm to implementing only linear pricing, that is we let the set of admissible pricing to be:

$$\mathcal{P}_{lin} := \left\{ t \mapsto p_0 + \sqrt{\varepsilon} K t \mid p_0 \in \mathbb{R}, K \in [-1, +1] \right\} \subset C^1[0, \infty).$$

Given the assumptions, the consumer will respond to $p \in \mathcal{P}_{lin}$ with learning strategy as given in Proposition 4 (and Remark 3 taking care of the boundary issues). Therefore, the firm only needs to determine the optimal (p_0, K) , and we denote the expected payoff by $\mathcal{V}^S(x; p_0, K)$. Considering linear pricing from the firm's perspective is not without loss of generality. Nevertheless, linear pricing suffices to fully answer our first main question, whether constant price is always optimal. By showing that the firm can improve its expected profit by increasing or decreasing the price linearly, we know that constant price is not generally optimal for the firm when it cannot track the consumer's belief evolution process. The most important qualitative property of the firm's optimal pricing strategy is the direction of price evolution. Linear pricing is general enough for us to see whether the price should stay constant, increase, or decrease over time, which is managerially relevant to firms in their pricing decision.

The discussion of linear pricing also serves as a template for understanding pricing strategies in more general settings where \mathcal{P} could include non-linear pricing strategies as long as Assumption 1 holds. With these assumptions, the consumer ε -optimal learning decision to any $p \in \mathcal{P}$ is entirely determined by the value of p_t and its slope p'_t at any given time taccording to Proposition 4 which falls under our linear pricing framework. In practice, due to some regulations \mathcal{P} could involve a restriction on how fast the firm can adjust the price over time. Moreover, suppose that the firm is sufficiently myopic (m >> 0), or if the product search is very informative $(\sigma^2 >> 0)$ so that the consumer's valuation diffuses and absorbed rapidly, or \mathcal{P} involves a restriction on the pricing function's second derivative. Then we can argue that any $p \in \mathcal{P}$ is approximately linear in the foreseeable future concerning the firm's ε -optimal profit. Therefore, by finding a local maximum (p_0^*, K^*) of $\mathcal{V}^S(x; ., .)$ with $K \sim 0$, given a sufficiently restrictive non-linear \mathcal{P} and certain parameters settings, (p_0^*, K^*) gives an initial price and slope of the ε -optimal pricing over \mathcal{P} for the firm to implement at t = 0. We shall return to elaborate further on this toward the end of this section. For t > 0 under such settings, the linear pricing payoff $\mathcal{V}^S(v; ., .)$ can be integrated over $v \sim$ distribution of the diffused valuation v_t , then computing the optimal pricing slope to continue evolving the ε -optimal strategy. Although a further analysis of the pricing dynamic based on this outline should be possible we shall leave such a challenging topic for future research.

We discuss two linear pricing cases. In the first case, the consumer has zero search costs (but still discounts the future). In the second case, the consumer has a positive search cost.

Zero Search Costs

When the consumer has zero search costs, the continuation value of searching is positive, whereas the payoff from quitting is zero. Therefore, she would never quit searching without purchasing the product. Equivalently, the consumer's quitting boundary is $-\infty$. The consumer's optimal search strategy only has a single boundary \bar{V}_t .

If the firm is perfectly patient, it will not have a direct incentive to speed up the consumer's decision-making process. A purchase at any time gives the firm the same payoff. Hence, it does not have a strong incentive to increase the price over time to push the consumer to make an early decision. In addition, the firm will charge a sufficiently high price such that the consumer's payoff from purchasing the product is negative initially. Therefore, discounting does not reduce the consumer surplus if she delays the decision by searching for more information. Even if the price does not decrease over time, the consumer will keep searching for information because she has nothing to lose. So, the firm has no incentive to reduce the price over time to prevent the consumer from quitting. In sum, in this case, the firm has little incentive to charge non-stationary prices. The following proposition shows that the optimal price is arbitrarily close to a constant price when the firm is perfectly patient. On the contrary, the firm charges non-stationary prices if it discounts the future.

Proposition 6. Suppose the search cost is zero c = 0.

When the firm is perfectly patient, m = 0, for any fixed initial price p_0 the firm can

approach the profit supremum

$$V^{S}(x) = \sup_{(p_{0},K)} \mathcal{V}^{S}(x;p_{0},K) = 2p_{0} + \frac{\sigma}{\sqrt{2r}} - g - x,$$

and, if p_0 is not fixed, then it is optimal for the firm to set p_0 as large as possible.

When the firm's discount factor is sufficiently small or sufficiently large, the slope K of the optimal linear pricing is bounded from zero.

Intuitively, since there's no exit boundary, a consumer started at any x will eventually purchase, and m = 0 means the firm can wait indefinitely. Therefore, it can charge an arbitrarily high price p_0 , consistent with the finding in Branco et al. (2012).

Positive Search Costs

The previous section shows $K \to 0$ if the search cost and the firm's discount factor are 0. For the slope of the optimal price to be bounded from zero, the firm must discount the future if there is no search cost. In this section, we consider the case with a positive search cost. In the presence of a positive search cost, the continuation value of searching may be negative. Hence, both the purchase and exit boundaries are finite, giving two unknowns to be determined at any given time in the optimal search problem.

Since the optimal price is non-stationary with K bounded from zero in the presence of search friction, even if the firm is perfectly patient, we will focus on the no-discounting case in this section.⁷ We consider K in the vicinity of 0, and examine whether the firm would benefit from slightly increasing ($K \gtrsim 0$) or decreasing ($K \lesssim 0$) the price over time. The consumer optimal response to the linear pricing $t \mapsto p_0 + \sqrt{\varepsilon}Kt$ is characterized by the moving purchase and exit boundaries \bar{V}_t , and \underline{V}_t as in Proposition 4, in particular we have

$$\bar{V}_0 = p_0 + \bar{V}[\sqrt{\varepsilon}K], \qquad \underline{V}_0 = p_0 + \underline{V}[\sqrt{\varepsilon}K],$$

where $\overline{V}[\sqrt{\varepsilon}K], \underline{V}[\sqrt{\varepsilon}K]$ depends on $\sqrt{\varepsilon}K$ and are determined in Proposition 4.

Proposition 7. Suppose the search cost is positive c > 0, and the firm is perfectly patient m = 0. The firm's expected profit from a consumer whose initial valuation is x is the following.

$$\mathcal{V}^{S}(x;p_{0},K) = \frac{p_{0} - g + (\bar{V}_{0} + x - 2\underline{V}_{0})}{1 - \exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^{2}}(\bar{V}_{0} - \underline{V}_{0})\right)} - \frac{2(\bar{V}_{0} - \underline{V}_{0})}{\left(1 - \exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^{2}}(\bar{V}_{0} - \underline{V}_{0})\right)\right)^{2}}$$

 $^{^{7}}$ As we can see from the zero search cost case, the firm is more inclined to charge non-stationary prices if it discounts the future.

$$-\frac{\left(p_{0}-g+\left(\bar{V}_{0}-x\right)\right)\exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^{2}}\left(x-\underline{V}_{0}\right)\right)}{1-\exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^{2}}\left(\bar{V}_{0}-\underline{V}_{0}\right)\right)}+\frac{2\left(\bar{V}_{0}-\underline{V}_{0}\right)\exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^{2}}\left(x-\underline{V}_{0}\right)\right)}{\left(1-\exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^{2}}\left(\bar{V}_{0}-\underline{V}_{0}\right)\right)\right)^{2}}$$
(30)

if $x \in (\underline{V}_0, \overline{V}_0)$ and $K \neq 0$, and $\mathcal{V}^S(x; p_0, K) = (p_0 - g) \left(\frac{x - \underline{V}_0}{\overline{V}_0 - \underline{V}_0}\right)$ if $x \in (\underline{V}_0, \overline{V}_0)$ and K = 0. $\mathcal{V}^S(x; p_0, K) = 0$ if $x \leq \underline{V}_0$. $\mathcal{V}^S(x; p_0, K) = p_0 - g$ if $x \geq \overline{V}_0$.

The firm's value function in the above proposition allows us to characterize under what conditions the firm intends to increase the price over time and under what conditions the firm seeks to decrease the price over time. By keeping $K \sim 0$ it is also automatically optimal to set p_0 to be the optimal static price according to Branco et al. (2012):

$$\hat{p} = \hat{p}(x) = \begin{cases} \frac{x+g-\underline{V}}{2}, & \underline{V} + g < x < 2\overline{V} - \underline{V} + g \\ x - \overline{V}, & x \ge 2\overline{V} - \underline{V} + g \end{cases}$$

Define $q = \frac{x - V - g}{2(\bar{V} - V)}$ as the initial relative position of the consumer between the purchasing and quitting boundaries under \hat{p} . It turns out that

$$\frac{\partial \mathcal{V}^S}{\partial K}(x; p_0 = \hat{p}, K = 0) = \frac{(\bar{V} - \underline{V})^2}{3\sigma^2} (1 - 2q)q(1 - q) - (\bar{S}q + \underline{S}(1 - q))q(1 - q)q(1 - q) - (\bar{S}q + \underline{S}(1 - q))q(1 - q)q(1 - q)q(1 - q)q(1 - q))q(1 - q)q(1 - q)q$$

is not identically zero for all $q \in [0, 1]$ and its sign depends only on σ^2/r , c/r and q. This shows that for a generic $q \in [0, 1]$ the optimal strategy (p_0^*, K^*) is such that K^* must be bounded away from 0 even for m = 0. The seller can immediately improve its expected profit by setting $K \gtrsim 0$ if $\frac{\partial \mathcal{V}^S}{\partial K}(q; p_0 = \hat{p}, K = 0) > 0$, and by setting $K \lesssim 0$ if $\frac{\partial \mathcal{V}^S}{\partial K}(q; p_0 = \hat{p}, K = 0) < 0$. We summarize the result in Figure 4.

We divide the figure into four regions.

I (Low incentive to search) When the information is too noisy (low σ^2), the search cost is too high (high c), or the consumer values little about the future (high r), the consumer has a low incentive to search for information. In such cases, the firm needs to give the consumer a high surplus to encourage her to search, which hurts its profit. So, it becomes more attractive for the firm to convince the consumer to purchase the product at the beginning, based on the initial valuation and the expected price trajectory. For any given initial price, by charging an increasing price over time, the firm lowers the purchasing threshold at the beginning by making it more desirable for the consumer to make an immediate decision. Compared with the stationary pricing strategy of charging a lower constant price, this non-stationary pricing strategy moves the purchasing threshold in the same direction (downwards) without sacrificing the profit conditional on purchase.



Figure 4: Region plots of seller's expected profit improvement direction of K in the vicinity of $K \sim 0$, using m = 0, with c/r = 1 for the left plot, and $\sigma^2/r = 1$ for the right plot.

- II (High-value consumer) When the consumer has a high initial valuation, she is too valuable to lose from the firm's perspective. Therefore, the firm wants to increase the purchasing probability in this case. Moreover, a high-value consumer can earn a positive payoff from purchasing immediately, which decreases over time because of discounting. Thus, the firm also wants the consumer to buy quickly. An increasing pricing strategy reduces the benefits of searching and encourages the consumer to purchase quickly and with a higher likelihood.
- III (Medium-value consumer) When the consumer has a moderate interest in the product, an increase in price does not suffice to convince the consumer to purchase quickly without learning much additional information. Instead, it reduces the benefit of searching because the consumer knows she has to pay a higher price if she learns positive things. Therefore, an increasing price will lead to a quick exit rather than a quick purchase.

The firm can benefit from reducing the price gradually in this case. A decreasing price helps the firm keep the consumer engaged in the search process even if she receives some negative information early on. It increases the purchasing likelihood. Because of the moderate initial valuation, the firm can still obtain a decent profit at a lower price. This pricing strategy protects the firm from missing potentially valuable consumers.

IV (Low-value consumer) By charging a decreasing price over time, the firm can keep the

consumer engaged in the search process even if she receives some negative information early on. However, it is not worth it for the firm to reduce the price over time for two reasons. First, the profit from an immediate purchase is already low when the consumer has a low initial valuation. The firm will obtain an even lower profit from purchasing if the consumer searches for a while and eventually buys at a lower price. Second, the consumer must accumulate a lot of positive information before purchasing due to the low initial valuation. The purchasing probability will still be low even if the price slightly reduces over time, and cannot offset the cost of a lower profit per purchase.

The firm quickly filters out many consumers by implementing an increasing pricing strategy instead. On the one hand, the loss from not converting these people is limited due to the low profit per purchase and the low purchasing probability. On the other hand, the benefits of charging a higher price to the remaining consumers are high. Any consumers not quitting despite the increasing price must have learned positive information and are more valuable to the firm.

So far, we discussed K in the vicinity of 0 by analyzing the derivative of $\mathcal{V}^{S}(x; p_{0}, K)$ at K = 0 with $p_{0} = \hat{p} = \frac{x+g-V}{2}$. For any $K \neq 0$, by solving $\frac{\partial \mathcal{V}^{S}}{\partial p_{0}}(x; p_{0}^{*}, K) = 0$ we find that the optimal initial price p_{0} that maximizes $\mathcal{V}^{S}(x; ..., K)$ is:

$$p_0^*(x,K) := \frac{x+g}{2} + \frac{\sigma^2}{2\sqrt{\varepsilon}K} - \frac{\underline{V}[\sqrt{\varepsilon}K]}{2} \left(1 - \coth\frac{\sqrt{\varepsilon}K}{\sigma^2} \left(\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K]\right)\right) \\ - \frac{\bar{V}[\sqrt{\varepsilon}K]}{2} \coth\frac{\sqrt{\varepsilon}K}{\sigma^2} \left(\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K]\right)$$

for $x \in (\underline{V}_0, \overline{V}_0)$ and we can check that $\lim_{K \to 0} p_0^*(x, K) = \hat{p}(x)$.

Lemma 5. Suppose that $\sigma\sqrt{r} > 0$, then there exists $\lambda_1 > 0$ sufficiently small such that if $(p_0^*, K^*) \in \mathbb{R} \times [-\lambda_1, +\lambda_1]$ is any local maximum point of $\mathcal{V}^S(x; ., .)$, then either $p_0^* < \hat{p}, K^* \gtrsim 0$, or $p_0^* > \hat{p}, K^* \lesssim 0$.

In fact, it is possible to find values of q, r, c, σ^2 , and $\lambda_1 > 0$ such that (p_0^*, K^*) is the unique (hence global) maximum point of $\mathcal{V}^S(x; ., .)$ over $\mathbb{R} \times [-\lambda_1, +\lambda_1]$, satisfying either $p_0^* < \hat{p}, K^* \gtrsim 0$, or $p_0^* > \hat{p}, K^* \lesssim 0$.

We note some implications of Lemma 5 beyond the linear pricing strategies. Suppose that we expand the seller's set of admissible pricing strategies to:

$$\mathcal{P}_{\lambda_1,\lambda_2} := \left\{ p \in W^{2,\infty}(\mathbb{R}_{>0}) \mid \sup_{t \in \mathbb{R}_{>0}} |p_t'| \le \lambda_1, \operatorname{ess\,sup}_{t \in \mathbb{R}_{>0}} |p_t''| \le \lambda_2 \right\}$$

for some $\lambda_1, \lambda_2 > 0$, and by setting $p_0 := \lim_{t\to 0+} p_t$ we interpret $\mathcal{P}_{\lambda_1,\lambda_2} \subset C^1[0,\infty)$. If $\lambda_2 = 0$. Then $\mathcal{P}_{\lambda_1,\lambda_2}$ reduces to the set of linear pricing strategies, however, when $\lambda_2 > 0$ we also allow strategies that are non–linear, but not *too* non–linear. If $\lambda_1 = 0$ then $\mathcal{P}_{\lambda_1,\lambda_2}$ contains only static prices. In practice, sellers may be restricted by some regulations in how fast they can change the price over time, which means $\lambda_1 > 0$ cannot be too large. On the other hand, for any $\delta > 0$ we have:

$$V^{S}(x) = \sup_{p \in \mathcal{P}_{\lambda_{1},\lambda_{2}}} \left[\mathbb{E} \left[e^{-m\tau} (p_{\tau} - g) \cdot \mathbf{1}_{v_{\tau} \ge p_{\tau}} \cdot \mathbf{1}_{\tau < \delta} | v_{0} = x \right] + e^{-m\delta} \int_{\mathbb{R}} \mathcal{V}^{S}(x; p_{\cdot + \delta}) U(\delta, x) dx \right].$$

$$(31)$$

By Assumption 1, we already have that the buyer is myopic and only cares about the p_t and p'_t at any given time t. Given a finite $\lambda_2 > 0$, we can find $\delta > 0$ such that the first term of (31) can be approximated with a linear pricing $l_t := p_0 + p'_0 \cdot t$ up to an order of some $\varepsilon > 0$, i.e. $\mathbb{E}[e^{-m\tau}(p_{\tau} - l_{\tau}) \cdot 1_{v_{\tau} \ge p_{\tau}} \cdot 1_{\tau < \delta} | v_0 = x] < \varepsilon/2$. Since survival probability satisfies $\mathbb{P}[\tau^*[l] > \delta] = O\left(\frac{1}{\sigma\sqrt{\delta}}\right)$, we have that the second term of (31) is of $\left(\frac{1}{\sigma\sqrt{\delta}}\right)$ -order. Suppose that $\sigma >> 0$, which is relevant in a situation where a large amount of information can be transferred to the buyer effectively, then we may argue that the second term of (31) is $\varepsilon/2$. In conclusion, for some sufficiently small $\lambda_1 > 0$ and sufficiently large $\sigma^2 >> 0$, Lemma 5 in fact classifies the initial value and initial slope of the optimal pricing strategy over $\mathcal{P}_{\lambda_1,\lambda_2}$.

4.3 Forced–Purchase Strategy

In the previous sections, we focused on the perturbative regime of pricing strategies: linear and slow-moving prices. However, we will show that, in a certain case, an alternative pricing strategy is tractable and leads to interesting results. Namely, when the buyer's initial valuation is sufficiently high, it is optimal for the seller to force an immediate purchase by increasing the price as sharply as possible. This presents another way for the firm to utilize non-stationary pricing to its advantage. In this section, we impose neither the myopic assumption nor the slow-varying price assumption from Assumption 1. The main result of this section is as follows.

Proposition 8. Let $h \in \mathcal{P}_T$ be an arbitrary pricing strategy strictly monotonically increasing over [0, T] with $h_0 = 0$ and let $p_0 \in \mathbb{R}$ be a constant. Then

$$\lim_{K \to \infty} \mathcal{V}^{S}(x; p_{0} + Kh) = \begin{cases} p_{0} - g, & \text{if } x > p_{0} \\ 0, & \text{if } x \le p_{0} \end{cases}.$$
(32)

Further, let $\tau^*[p] \in \mathcal{T}$ be the ε -optimal buyer's stopping time to the pricing strategy $p \in \mathcal{P}_T$.

Then for the given parameters m, r, c, σ^2 such that $\underline{V} > -\infty$ (i.e. c > 0), there exists $\overline{x} := \overline{x}(m, r, c, \sigma^2)$ such that if $x > g + \overline{x}$ then

$$V^{S}(x) = \sup_{p \in \mathcal{P}_{T}} \mathcal{V}^{S}(x; \tau^{*}[p], p) = x - g$$

can be approached by the sequence

$$\{p_n := p_{0,n} + K_n h \in \mathcal{P}_T\}_{n \in \mathbb{Z}_{\ge 0}}$$

$$\tag{33}$$

where $p_{0,n} \to x-$ and $K_n \to +\infty$.

We note that for $x > g + \bar{x}(m, r, c, \sigma^2)$, c > 0, we have $V^S(x) = x - g$ regardless of how large T > 0 is chosen. The condition c > 0 is important as we recall the c = 0, m = 0 case with $T \to \infty$ from §4.2 that the optimal initial price is unbounded, leading to $V^S(x) = \infty$. Even if we require $p_0 = x$ -, the optimal linear pricing strategy is to set the slope $K \gtrsim 0$ as close to 0 as possible and achieve $\mathcal{V}^S(x; p_0 = x - K) \sim x - g + \frac{\sigma}{\sqrt{2r}} > x - g$. Although the second inequality in (52) continues to hold even if $\underline{V}_t[p] = -\infty < g$, since the buyer never exits without the exit boundary, the real reason the seller can achieve an expected payoff higher than x - g is because Martingale stopping theorem fails as $v_{t \wedge \tau^*[p]}$ is not bounded in this case.

It is also interesting to compare the result in this section with the result under a constant price, where the optimal constant price is $x - \overline{V}$ for all $x \ge 2\overline{V} - \underline{V} + g$. The seller's profit under the optimal constant price, $x-g-\overline{V}$, is lower than the profit under the forced-purchase strategy in this section, x - g. Intuitively, the highest constant price the seller can charge to induce immediate purchase is $p_0 + \overline{V}$. By charging an increasing price over time, the seller reduces the consumer's incentive to search and encourages the consumer to make a purchase decision more easily. Therefore, the consumer will be willing to purchase immediately at a higher initial price. By using non-stationary pricing, the seller is able to increase its profit by \overline{V} .

5 Conclusion

This paper introduces a novel framework where firms adopt non-stationary pricing strategies. Our finding challenges the conventional reliance on stationary pricing, and shows that non-stationary pricing strategies can outperform stationary ones. Our first contribution is to provide a theoretical advance in optimal control by incorporating non-stationary strategies into a consumer search framework. Unlike previous works, the non-stationarity in our model arises endogenously from firms' strategic pricing in response to consumer gradual learning. Our second contribution is to offer practical insights into how firms can proactively adjust their pricing strategies to adapt to privacy regulations.

Appendix

Proof of Proposition 1. Part 1:

Given any learning strategy $\tau \in \mathcal{T}$, it is clear that $\mathcal{V}^B(0, x; \tau, \tilde{p}) \leq \mathcal{V}^B(0, x; \tau, p)$ for all $x \in \mathbb{R}$, therefore: $V^B(0, x; \tilde{p}) \leq V^B(0, x; p)$. Meanwhile, we have $\max\{x - \tilde{p}_0, 0\} = \max\{x - p_0, 0\}$ from the assumption that $h_0 = 0$. It follows that:

$$\begin{split} \bar{V}_0[\tilde{p}] &= \sup \left\{ x \in \mathbb{R} \mid V^B(0, x; \tilde{p}) > \max\{x - p_0, 0\} \right\} \\ &\leq \sup \left\{ x \in \mathbb{R} \mid V^B(0, x; p) > \max\{x - p_0, 0\} \right\} = \bar{V}_0[p] \end{split}$$

and similarly:

$$\underline{V}_{0}[\tilde{p}] = \inf \left\{ x \in \mathbb{R} \mid V^{B}(0, x; \tilde{p}) > \max\{x - p_{0}, 0\} \right\}$$

$$\geq \inf \left\{ x \in \mathbb{R} \mid V^{B}(0, x; p) > \max\{x - p_{0}, 0\} \right\} = \underline{V}_{0}[p],$$

proving the claim.

Part 2:

Without the loss of generality, let's only consider t = 0 and h such that $h_0 = 0$, we can always redefine t and shift \tilde{p} by constant otherwise. Moreover, any strategies represented by $\bar{V}[p], \underline{V}[p] : \mathbb{R} \to \mathbb{R}$ will give an identical payoff to the strategy given by $\min\{\bar{V}[p], \bar{v}\}, \max\{\underline{V}[p], \underline{v}\}$. Therefore, it is without the loss of generality to assume $\bar{V}[p], \underline{V}[p] \in [\underline{v}, \bar{v}]$.

Suppose that $\bigcap_{K>0}(\underline{V}_0[\tilde{p}], \overline{V}_0[\tilde{p}])$ is open and containing x, hence $V^B(t = 0, x; \tilde{p}) > 0$ for all K > 0. We can find $\tau_{x,\varepsilon} \in \mathcal{T}$ such that $\mathcal{V}^B(0, x; \tau_{x,\varepsilon}, \tilde{p}) \geq V^B(0, x; \tilde{p}) - \varepsilon$ and for any $\varepsilon' > 0$ we can find $\delta > 0$ such that $\mathbb{P}[\tau_{x,\varepsilon} < \delta] < \varepsilon'$ for all K > 0. Additionally, from the first part we know that $\overline{V}_0[p] \geq \overline{V}_0[\tilde{p}]$. Then:

$$V^{B}(0,x;\tilde{p}) \leq \mathcal{V}^{B}(0,x;\tau_{x,\varepsilon},\tilde{p}) + \varepsilon$$

$$\leq (1-\varepsilon')e^{-r\delta}\mathbb{E}\left[\max\left\{\bar{V}_{\tau_{x,\varepsilon}}[p] - p_{\tau_{x,\varepsilon}} - Kh_{\tau_{x,\varepsilon}},0\right\} | \tau_{x,\varepsilon} \geq \delta\right] + \varepsilon'\bar{V}_{0}[p] + \varepsilon.$$

Since ε' and ε are arbitrarily small, while the first term is zero for all sufficiently large K >> 0, we have that $V^B(0, x; \tilde{p}) \leq 0$, a contradiction. Therefore, $\bigcap_{K>0}(\underline{V}_0[\tilde{p}], \overline{V}_0[\tilde{p}])$ contains no open sets, proving $\lim_{K\to+\infty} (\overline{V}_0[\tilde{p}] - \underline{V}_0[\tilde{p}]) = 0$. In other words, for any $x \neq \tilde{p}_0$ there exists K >> 0 sufficiently large such that the continue learning option is sub–optimal and the buyer immediately purchase if $x > \tilde{p}_0$ and exit if $x < \tilde{p}_0$, proving $\overline{V}_0[\tilde{p}] \searrow \max\{\tilde{p}_0, \underline{v}\}, \underline{V}_0[\tilde{p}] \nearrow$ $\min\{\tilde{p}_0, \bar{v}\}$ as $K \to +\infty$. Suppose that K < 0, consider any $x \in \mathbb{R}$, we note that

$$V^B(0,x;\tilde{p}) \ge \mathcal{V}^B(0,x;\delta,\tilde{p}) \ge e^{-r\delta} \mathbb{E}\left[\max\left\{v_{\delta}^x - p_{\delta} - Kh_{\delta}, 0\right\}\right] - c\delta \ge -e^{-r\delta}(Kh_{\delta} + p_{\delta}) - c\delta$$

where δ denotes the simple policy of stopping exactly at some time $\delta > 0$ regardless of the valuation, and the first inequality followed from the sub-optimality of δ . Therefore, for all sufficiently negative $K \ll 0$, we have $V^B(0, x; \tilde{p}) > 0$, thus it is optimal to continue searching: i.e. $\underline{V}_0[\tilde{p}] \ll x \ll \overline{V}_0[\tilde{p}]$ for any $x \in [\underline{v}, \overline{v}]$, proving $\overline{V}_0[\tilde{p}] \nearrow \overline{v}$ and $\underline{V}_0[\tilde{p}] \searrow \underline{v}$ as $K \to -\infty$.

Proof of Lemma 4. Consider a fixed $(t,x) \in [0,T] \times \mathbb{R}$, and suppose that $V^B(t,x;q) \leq V^B(t,x;p)$. For an arbitrary $\varepsilon > 0$, let $\tau_{t,x,\varepsilon}[p] \in \mathcal{T}$ be such that $\mathcal{V}^B(t,x;\tau_{t,x,\varepsilon}[p],p) \geq V^B(t,x;p) - \varepsilon$, then

$$\begin{aligned} V^B(t,x;q) &\geq \mathcal{V}^B(t,x;\tau_{t,x,\varepsilon}[p],q) > \mathcal{V}^B(t,x;\tau_{t,x,\varepsilon}[p];p) - \max_{s\in[t,T]} e^{-r(s-t)} |p_s - q_s| \\ &\geq V^B(t,x;p) - \max_{s\in[t,T]} e^{-r(s-t)} |p_s - q_s| - \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it must be the case that:

$$V^B(t,x;p) \ge V^B(t,x;q) \ge V^B(t,x;p) - \max_{s \in [t,T]} e^{-r(s-t)} |p_s - q_s|$$

If $V^B(t, x; q) \ge V^B(t, x; p)$, then we simply switch the role of p, q and follow through with the above argument, hence we get that

$$|V^B(t,x;p) - V^B(t,x;q)| \le \max_{s \in [t,T]} e^{-r(s-t)} |p_s - q_s|,$$

which proves the result.

Proof of Proposition 2. We note that $V^B(t, x - \sqrt{\varepsilon}h_t; p)$ is simply the solution $V^B(t, x; p)$ shifted according to $\sqrt{\varepsilon}Kh$ which satisfies the value-matching and smooth-pasting conditions at $\bar{V}[p] + \sqrt{\varepsilon}h$ and $\underline{V}[p] + \sqrt{\varepsilon}h$, but does not satisfies the PDE, hence the $\sqrt{\varepsilon}V_1^B$ correction is needed. By adding $\sqrt{\varepsilon}V_1^B$ correction, we further need a $\sqrt{\varepsilon}$ -order correction to the purchase and exit boundaries $\bar{V}[p] + \sqrt{\varepsilon}h$ and $\underline{V}[p] + \sqrt{\varepsilon}h$ which take the form (23). We find the equation for V_1^B by substituting the ansatz (21) into the PDE for $V^B(.,.;\tilde{p})$ and collecting the $\sqrt{\varepsilon}$ -order terms:

$$\frac{\sigma(x)^2}{2}\partial_x^2 V_1^B(t,x) + \partial_t V_1^B(t,x) - rV_1^B(t,x)$$

$$-h'_t \partial_x V^B(t,x;p) + h_t \sigma(x) \sigma'(x) \partial_x^2 V^B(t,x;p) = 0.$$
 (34)

To study \overline{R} and \underline{R} we analyze the boundary conditions of $V^B(.,.;\tilde{p})$ to the first-order in $\sqrt{\varepsilon}$. Note that $V^B(t, x - \sqrt{\varepsilon}h_t; p)$ automatically satisfies the value-matching conditions at $\overline{V}[\tilde{p}]$ and $\underline{V}[\tilde{p}]$, as we will confirm below, because $\partial_x V^B(t, \overline{V}_t[p]; p) = 1$ and $\partial_x V^B(t, \underline{V}_t[p]; p) = 0$. We have by substituting the ansatz (21) and (26) into the boundary conditions and comparing the $\sqrt{\varepsilon}$ -order terms:

$$V^{B}(t, \bar{V}_{t}[\tilde{p}]; \tilde{p}) = \bar{V}_{t}[\tilde{p}] - \tilde{p}_{t}$$

$$\implies V^{B}(t, \bar{V}_{t}[p] + \sqrt{\varepsilon}\bar{R}_{t}; p) + \sqrt{\varepsilon}V_{1}^{B}(t, \bar{V}_{t}[p]) = \bar{V}_{t}[p] - p_{t} + \sqrt{\varepsilon}\bar{R}_{t}$$

$$V_{1}^{B}(t, \bar{V}_{t}[p]) = -\bar{R}_{t}\partial_{x}V^{B}(t, \bar{V}_{t}[p]; p) + \bar{R}_{t} \implies V_{1}^{B}(t, \bar{V}_{t}[p]) = 0. \quad (35)$$

$$\partial_x V^B(t, \bar{V}_t[\tilde{p}]; \tilde{p}) = 1 \implies \partial_x V^B(t, \bar{V}_t[p] + \sqrt{\varepsilon}\bar{R}_t; p) + \sqrt{\varepsilon}\partial_x V_1^B(t, \bar{V}_t[p]) = 1$$

$$\partial_x V_1^B(t, \bar{V}_t[p]) = -\bar{R}_t \partial_x^2 V^B(t, \bar{V}_t[p]; p) \implies \bar{R}_t = -\frac{\partial_x V_1^B(t, \bar{V}_t[p])}{\partial_x^2 V^B(t, \bar{V}_t[p]; p)}.$$
 (36)

$$V^{B}(t, \underline{V}_{t}[\tilde{p}]; \tilde{p}) = 0 \implies V^{B}(t, \underline{V}_{t}[p] + \sqrt{\varepsilon}\underline{R}_{t}; p) + \sqrt{\varepsilon}V_{1}^{B}(t, \underline{V}_{t}[p]) = 0 \qquad \Longrightarrow V_{1}^{B}(t, \underline{V}_{t}[p]) = 0.$$
(37)

$$\partial_x V^B(t, \underline{V}_t[\tilde{p}]; \tilde{p}) = 0 \implies \partial_x V^B(t, \underline{V}_t[p] + \sqrt{\varepsilon}\underline{R}_t; p) + \sqrt{\varepsilon}\partial_x V_1^B(t, \underline{V}_t[p]) = 0$$

$$\partial_x V_1^B(t, \underline{V}_t[p]) = -\underline{R}_t \partial_x^2 V^B(t, \underline{V}_t[p]; p) \implies \underline{R}_t = -\frac{\partial_x V_1^B(t, \underline{V}_t[p])}{\partial_x^2 V^B(t, \underline{V}_t[p]; p)}.$$
 (38)

From $V^B(T, x; \tilde{p}) = V_0^B(x; \tilde{p}_T)$ we have that $V_1^B(T, x) = 0$. We recognize the PDE (34) with (35) and (37) as a backward parabolic (fixed) boundary-value problem. We may transform the problem into the more standard parabolic form for: $\tilde{V}_1^B(t', x') :=$ $V_1^B(T - t', \underline{V}_{T-t'}[p] + (\bar{V}_{T-t'}[p] - \underline{V}_{T-t'}[p])x')$ on $\tilde{\Omega} := [0, \infty) \times [0, 1]$ with smooth coefficients $(a_{ij}(.), b_i(.), c(.))$, according to our smoothness assumptions on $\bar{V}[p], \underline{V}[p]$, and $\sigma(.)$. Since $-h'_t \partial_x V^B(., .; p) + h_t \sigma(.) \sigma'(.) \partial_x^2 V^B(., .; p)$ is assumed smooth on $\tilde{\Omega}$, we can apply (Evans, 2022, Chapter 7, Theorem 7) or (Friedman, 2008, Chapter 3, §5, Corollary 2) to conclude the existence of the smooth solution \tilde{V}_1^B to the parabolic initial boundary-value problem. Transform back to the original problem, we get the smooth V_1^B . The solution is unique, and admits a probabilistic expression (22) via the semi-elliptic version of Feynman–Kac formula (Øksendal, 2003, Theorem 9.1.1).

Suppose that $h := K\tilde{h}$, where $\tilde{h} \in \mathcal{P}_T$ is monotonically increasing in t, and that $\sigma'(.) = O(\varepsilon)$. We define $\bar{S} := \bar{R}/K : \mathbb{R} \to \mathbb{R}$, and $\underline{S} := \underline{R}/K : \mathbb{R} \to \mathbb{R}$. It remains to show that $\bar{S}_t \leq 0$ and $\underline{S}_t \geq 0$. From our assumption $\sigma'(.) = O(\varepsilon)$ so we may ignore the second term in the $\sqrt{\varepsilon}$ -order equation (22). Since $V^B(t, .; p)$ is monotonically increasing in x from Lemma 1, it follows from (22) that $V_1^B(t, x)/K \leq 0$ for any $(t, x) \in \Omega$. In particular, $\partial_x V_1^B(t, \bar{V}_t[p])/K \geq 0$ and $\partial_x V_1^B(t, \underline{V}_t[p])/K \leq 0$.

Now, let us show that $\partial_x^2 V^B(t, \bar{V}_t[p]; p) \ge 0$. Let $\mathbf{x}_0 = (t, \bar{V}_t[p]) \in \partial\Omega$ be a point on the purchase boundary, then we can find sequences $\{\mathbf{x}_i^+ = (t_i, x_i^+)\}_{i=0}^{\infty}$ and $\{\mathbf{x}_i^- = (t_i, x_i^-)\}_{i=0}^{\infty} \subset \Omega$ converging to \mathbf{x}_0 such that $x_i^- \le \bar{V}_{t_i}[p] \le x_i^+$ for all $i \ge 0$. Since $V^B(.,.;p)$ is the viscosity solution, we have $c + rV^B(\mathbf{x}_i^+;p) - \partial_t V^B(\mathbf{x}_i^+;p) - \frac{\sigma(\mathbf{x}_i^+)^2}{2} \partial_x^2 V^B(\mathbf{x}_i^+;p) \ge 0$, while $c + rV^B(\mathbf{x}_i^-;p) - \partial_t V^B(\mathbf{x}_i^-;p) - \frac{\sigma(\mathbf{x}_i^-)^2}{2} \partial_x^2 V^B(\mathbf{x}_i^-;p) = 0$ for all $i \ge 0$. But $V^B(\mathbf{x}_i^+;p) = x_i^+ - p_{t_i}$, so $\frac{\sigma(\mathbf{x}_i^+)^2}{2} \partial_x^2 V^B(\mathbf{x}_i^+;p) = 0$, hence it follows from the continuous differentiability of $V^B(.,.;p)$ across the boundary $\partial\Omega$ that $\partial_x^2 V^B(t, \bar{V}_t[p];p) \ge 0$. Similarly, we have that $\partial_x^2 V^B(t, V_t[p];p) \ge 0$.

It follows from (36) and (38) that the sign of \bar{S}_t and \underline{S}_t are the opposite as the sign of $\partial_x V_1^B(t, \bar{V}_t[p])/K$ and $\partial_x V_1^B(t, \underline{V}_t[p])/K$, respectively, therefore, we have $\bar{S}_t \leq 0$ and $\underline{S}_t \geq 0$ for $t \in \mathbb{R}$ as claimed.

Proof of Proposition 3. Let $p^T, l_{\mathbf{x}}^T \in \mathcal{P}_T$ be given by some pricing strategies which coincide with $p, l_{\mathbf{x}}$ over $[0, T - \varepsilon]$ and constant for all $t \geq T$. By Lemma 4, we have $|V^B(t, x; p^T) - V^B(t, x; l_{\mathbf{x}}^T)| \leq \max_{s \in [t,T]} e^{-r(s-t)} |p_s - l_{\mathbf{x},s}| \leq \max_{s \geq t} e^{-r(s-t)} |p_s - l_{\mathbf{x},s}|$. Since this inequality holds for all T, we conclude that $|V^B(t, x; p) - V^B(t, x; l_{\mathbf{x}})| \leq \max_{s \geq t} e^{-r(s-t)} |p_s - l_{\mathbf{x},s}|$. But from Taylor's Theorem, we have $|p_s - l_{\mathbf{x},s}| \leq \frac{M}{2}(s-t)^2$ for all $s \geq t$. It follows that $\max_{s \geq t} e^{-r(s-t)} |p_s - l_{\mathbf{x},s}| \leq \frac{M}{2} \max_{s \geq t} (s-t)^2 e^{-r(s-t)} = \frac{2M}{r^2} e^{-2}$. Therefore, $|V^B(t, x; p) - V^B(t, x; l_{\mathbf{x}})| < \varepsilon$ if $r > e^{-1} \sqrt{2M/\varepsilon}$.

Proof of Proposition 4. In the special case of linear pricing $t \mapsto p_t := p_0 + \sqrt{\varepsilon}Kt$ the value function takes the form (25) over Ω as we can directly check that it satisfies the PDE of (20). Let's define $K_{\pm} := \frac{\sqrt{\varepsilon}K \pm \sqrt{\varepsilon}K^2 + 2r\sigma^2}{\sigma^2}$ for convenience. The purchase and exit boundaries ansatz take the form (26). We determine the unknown $A_1, A_2, \overline{V}[\sqrt{\varepsilon}K]$, and $\underline{V}[\sqrt{\varepsilon}K]$ from the boundary conditions

$$V^{B}(t,\bar{V}_{t}) = \bar{V}_{t} - p_{t} \implies A_{1}e^{K_{-}\bar{V}[\sqrt{\varepsilon}K]} + A_{2}e^{K_{+}\bar{V}[\sqrt{\varepsilon}K]} - \frac{c}{r} = \bar{V}[\sqrt{\varepsilon}K]$$
(39)

$$\partial_x V^B(t, \bar{V}_t) = 1 \implies A_1 K_- e^{K_- \bar{V}[\sqrt{\varepsilon}K]} + A_2 K_+ e^{K_+ \bar{V}[\sqrt{\varepsilon}K]} = 1$$

$$\tag{40}$$

$$V^{B}(t, \underline{V}_{t}) = 0 \implies A_{1}e^{K_{-}\underline{V}[\sqrt{\varepsilon}K]} + A_{2}e^{K_{+}\underline{V}[\sqrt{\varepsilon}K]} - \frac{c}{r} = 0$$

$$\tag{41}$$

$$\partial_x V^B(t, \bar{V}_t) = 0 \implies A_1 K_- e^{K_- \underline{V}[\sqrt{\varepsilon}K]} + A_2 K_+ e^{K_+ \underline{V}[\sqrt{\varepsilon}K]} = 0$$
(42)

From (41) and (42) we find that

$$A_1 = \frac{c}{r} \left(\frac{K_+}{K_+ - K_-} \right) e^{-K_- \underline{V}[\sqrt{\varepsilon}K]}, \quad A_2 = \frac{c}{r} \left(\frac{K_-}{K_- - K_+} \right) e^{-K_+ \underline{V}[\sqrt{\varepsilon}K]}.$$
(43)

Substituting (43) back into (40), we obtain the equation to be solved for $(\overline{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K])$:

$$e^{K_{+}(\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K])} - e^{K_{-}(\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K])} = \frac{r}{c} \cdot \frac{K_{-} - K_{+}}{K_{-}K_{+}},\tag{44}$$

we note that the LHS is an increasing function, hence the solution always exists. Finally, we find $\bar{V}[\sqrt{\varepsilon}K]$ by substituting (43) back into (39) and simplify:

$$\bar{V}[\sqrt{\varepsilon}K] = \frac{1}{K_{-}} + \frac{c}{r} \left(e^{K_{+}(\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K])} - 1 \right)$$
(45)

from this it is simple to find $\underline{V}[\sqrt{\varepsilon}K]$. Equation (44) and (45) is equivalent to the following non–linear system of equations:

$$\begin{cases} e^{\frac{\sqrt{\varepsilon}K + \sqrt{\varepsilon}K^2 + 2r\sigma^2}{\sigma^2}(\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K])} - e^{\frac{\sqrt{\varepsilon}K - \sqrt{\varepsilon}K^2 + 2r\sigma^2}{\sigma^2}(\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K])} &= \frac{\sqrt{\varepsilon}K^2 + 2r\sigma^2}{c} \\ \frac{c}{r} \left(e^{\frac{\sqrt{\varepsilon}K + \sqrt{\varepsilon}K^2 + 2r\sigma^2}{\sigma^2}(\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K])} - 1 \right) - \bar{V}[\sqrt{\varepsilon}K] &= \frac{\sqrt{\varepsilon}K + \sqrt{\varepsilon}K^2 + 2r\sigma^2}{2r} \end{cases}$$

$$(46)$$

When $\sqrt{\varepsilon}K \sim 0$, we may obtain a simple expression for $\overline{V}[\sqrt{\varepsilon}K]$ and $\underline{V}[\sqrt{\varepsilon}K]$ to the ε -order. We substituting the ansatz (27) into (44), (45), and comparing the zeroth-order and $\sqrt{\varepsilon}$ -order terms we get the claimed expression for $\overline{S} := \overline{R}/K, \underline{S} := \underline{R}/K$. The signs of \overline{S} and \underline{S} followed from the Proposition 2, but one can also verify explicitly. \Box

Proof of Proposition 6. From the first equation of (46), when $c \to 0+$, the RHS becomes large which means $\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K]$ becomes large, and the LHS is $\sim e^{\frac{\sqrt{\varepsilon}K+\sqrt{\varepsilon}K^2+2r\sigma^2}{\sigma^2}(\bar{V}[\sqrt{\varepsilon}K]-\underline{V}[\sqrt{\varepsilon}K])}$. Therefore, the second equation of (46) together with (26) gives:

$$\bar{V}_t = p_0 + \sqrt{\varepsilon}Kt + \frac{\sqrt{\varepsilon}K^2 + 2r\sigma^2 - \sqrt{\varepsilon}K}{2r}.$$

and $\underline{V}_t = -\infty$. Therefore, we only have one linearly moving boundary \overline{V}_t . Let's assume throughout also that $p_0 \geq g$. The solution U(t, v) to the heat equation with the single linearly moving absorbing boundary with initial condition $U(t = 0, v) = \delta(v - x), x \leq \overline{V}_0$, is

well-known:

$$U(t,v) = \frac{\exp\left(-\frac{\sqrt{\varepsilon}K}{\sigma^2}(v-x-\sqrt{\varepsilon}Kt)-\frac{\varepsilon K^2}{2\sigma^2}t\right)}{\sigma\sqrt{2\pi t}} \left(e^{-\frac{(v-\sqrt{\varepsilon}Kt-x)^2}{2t\sigma^2}}-e^{-\frac{(v-\sqrt{\varepsilon}Kt+x-2\bar{v}_0)^2}{2t\sigma^2}}\right).$$

Therefore, the purchase probability flux is:

$$-\frac{\sigma^2}{2}\partial_v U(t,\bar{V}_t) = \frac{\bar{V}_0 - x}{\sigma\sqrt{2\pi t^3}} \exp\left(-\frac{\left(\bar{V}_t - x\right)^2}{2t\sigma^2}\right)$$

It is now straightforward to compute the expected firm's payoff at t = 0:

$$\mathcal{V}^{S}(x;p_{0},K) := -\frac{\sigma^{2}}{2} \int_{0}^{\infty} e^{-ms}(p_{s}-g)\partial_{v}U(s,\bar{V}_{s})ds$$

$$= \left(p_{0}-g + \frac{\sqrt{\varepsilon}K}{\sqrt{2m\sigma^{2}+\varepsilon K^{2}}} \left(p_{0}-x + \frac{\sqrt{\varepsilon K^{2}+2r\sigma^{2}}-\sqrt{\varepsilon}K}{2r}\right)\right)$$

$$\times \exp\left(-\left(\frac{\sqrt{\varepsilon}K + \sqrt{2m\sigma^{2}+\varepsilon K^{2}}}{\sigma^{2}}\right) \left(p_{0}-x + \frac{\sqrt{\varepsilon K^{2}+2r\sigma^{2}}-\sqrt{\varepsilon}K}{2r}\right)\right), \quad (47)$$

for $x \leq \overline{V}_0$, otherwise if $x > \overline{V}_0$ then we have $\mathcal{V}^S(x; p_0, K) = p_0 - g$. In the special case where m = 0, we have

$$\mathcal{V}^{S}(x;p_{0},K) = \begin{cases} \left(2p_{0}-g-x+\frac{\sqrt{\varepsilon K^{2}+2r\sigma^{2}}-\sqrt{\varepsilon}K}{2r}\right) \\ \times \exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^{2}}\left(p_{0}-x+\frac{\sqrt{\varepsilon K^{2}+2r\sigma^{2}}-\sqrt{\varepsilon}K}{2r}\right)\right), & \sqrt{\varepsilon}K > 0 \\ p_{0}-g, & \sqrt{\varepsilon}K = 0 \\ x-g-\frac{\sqrt{\varepsilon K^{2}+2r\sigma^{2}}-\sqrt{\varepsilon}K}{2r}, & \sqrt{\varepsilon}K < 0 \end{cases}$$

For any fixed p_0 , we can approach the supremum $2p_0 + \frac{\sigma}{\sqrt{2r}} - g - x \ge p_0 - g$ of \mathcal{V}^S by choosing $\sqrt{\varepsilon}K \gtrsim 0$ as close to 0 as possible, and earning an extra of $\left(2p_0 + \frac{\sigma}{\sqrt{2r}} - g - x\right) - (p_0 - g) = p_0 - x + \frac{\sigma}{\sqrt{2r}}$. If we can also vary the initial price p, then it is optimal to set p as large as possible, i.e. the optimal price is unbounded.

For m > 0, the optimal K is now bounded from 0. This can also be seen for a general p by computing:

$$\frac{\partial \mathcal{V}^S}{\partial K}(x; p_0, K=0) = e^{-\frac{\sqrt{2m}}{\sigma} \left(p_0 - x + \frac{\sigma}{\sqrt{2r}}\right)}$$

$$\times \left(\frac{p_0 - x + \sigma/\sqrt{2r}}{\sigma\sqrt{2m}} - (p_0 - g)\left(\frac{p_0 - x + \sigma/\sqrt{2r} - (\sigma/r)\sqrt{m/2}}{\sigma^2}\right)\right),$$

we can see that this is always > 0 for sufficiently small and sufficiently large m > 0.

Proof of Proposition 7. The standard solution U_0 to the heat equation (28) with 2 absorbing non-moving boundaries at $\bar{V}_0 := p_0 + \bar{V}[\sqrt{\varepsilon}K], \underline{V}_0 := p_0 + \underline{V}[\sqrt{\varepsilon}K]$, and the initial condition $U_0(0, v) = \delta(v - x)$ is given by Karatzas and Shreve (2012):

$$U_0(t,v) = \frac{1}{\sigma\sqrt{2\pi t}} \sum_{k=-\infty}^{+\infty} \left[e^{-\frac{(v-x+2k(\bar{V}_0-\underline{V}_0))^2}{2t\sigma^2}} - e^{-\frac{(v+x-2\underline{V}_0+2k(\bar{V}_0-\underline{V}_0))^2}{2t\sigma^2}} \right].$$
 (48)

Equivalently:

$$U_0(t, v)dv = \mathbb{P}\left[x + \sigma W_t \in dv, \underline{V}_0 < x + \sigma W_s < \overline{V}_0, s \in [0, t]\right].$$

Instead of moving the boundary according to $\sqrt{\varepsilon}Kt$ we may consider the consumer valuation process to be the Brownian process with drift starting at x: $\tilde{v}_t = x - \sqrt{\varepsilon}Kt + \sigma W_t$ with fixed absorbing boundaries at $\bar{V}_0, \underline{V}_0$. If $\{W_t\}$ is the standard Brownian process on $(\Omega, \mathscr{F}, \Sigma, \mathbb{P})$ then $\{x + \sigma W_t\}$ is the Brownian process with drift starting at x, i.e. $\{\tilde{v}_t\}$ on $(\Omega, \mathscr{F}, \Sigma, \mathbb{Q})$ where

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathscr{F}_t} = \exp\left(-\frac{\sqrt{\varepsilon}K}{\sigma}W_t - \frac{\varepsilon K^2}{2\sigma^2}t\right).$$

Consequently, we have that the solution U to the heat equation (28) with moving boundaries $\bar{V}_t, \underline{V}_t$ is given by

$$\begin{split} U(t,v)dv &= \mathbb{P}\left[\tilde{v}_t \in v - \sqrt{\varepsilon}Kt + dv, \underline{V}_0 < \tilde{v}_s < \bar{V}_0, s \in [0,t]\right] \\ &= \mathbb{Q}\left[x + \sigma W_t \in v - \sqrt{\varepsilon}Kt + dv, \underline{V}_0 < x + \sigma W_s < \bar{V}_0, s \in [0,t]\right] \\ &= \exp\left(-\frac{\sqrt{\varepsilon}K}{\sigma^2}(v - x - \sqrt{\varepsilon}Kt) - \frac{\varepsilon K^2}{2\sigma^2}t\right) U_0(t, v - \sqrt{\varepsilon}Kt)dv \end{split}$$

Therefore, the purchase probability flux is:

$$-\frac{\sigma^2}{2}\partial_v U(t,\bar{V}_t) = \sum_{k=-\infty}^{+\infty} \frac{(2k+1)(\bar{V}_0 - \underline{V}_0) - (x - \underline{V}_0)}{\sigma\sqrt{2\pi t^3}} e^{\frac{2\sqrt{\varepsilon}Kk}{\sigma^2}(\bar{V}_0 - \underline{V}_0)} e^{-\frac{((2k+1)(\bar{V}_0 - \underline{V}_0) - (x - \underline{V}_0) + \sqrt{\varepsilon}Kt)^2}{2t\sigma^2}}$$
(49)

The term-by-term differentiation is justified at $v = \bar{V}_t$ for any fixed $x \in (\underline{V}_0, \bar{V}_0)$ because $0 < |\bar{V}_0 - x| < |\bar{V}_0 - \underline{V}_0|$, hence the series representation of $U_0(t, v - \sqrt{\varepsilon}Kt)$, and the derivative series both converge absolutely and uniformly for all v in some neighborhoods of

 \bar{V}_t and $t \in [0, \infty)$. We now compute the seller's expected profit:

Claim 1. The seller's expected profit from the consumer initially at $x \in (\underline{V}_0, \overline{V}_0)$ is:

$$\mathcal{V}^{S}(x;p_{0},K) = \frac{\left(p_{0} - g - \frac{\sqrt{\varepsilon}K}{\sqrt{2m\sigma^{2} + \varepsilon}K^{2}}(\bar{V}_{0} + x - 2\underline{V}_{0})\right)e^{-\frac{\sqrt{\varepsilon}K + \sqrt{2m\sigma^{2} + \varepsilon}K^{2}}{\sigma^{2}}(\bar{V}_{0} - x)}}{1 - e^{-\frac{2\sqrt{2m\sigma^{2} + \varepsilon}K^{2}}{\sigma^{2}}(\bar{V}_{0} - \underline{V}_{0})}} + \frac{\frac{2\sqrt{\varepsilon}K}{\sqrt{2m\sigma^{2} + \varepsilon}K^{2}}(\bar{V}_{0} - \underline{V}_{0})e^{-\frac{\sqrt{\varepsilon}K + \sqrt{2m\sigma^{2} + \varepsilon}K^{2}}{\sigma^{2}}(\bar{V}_{0} - x)}}{\left(1 - e^{-\frac{2\sqrt{2m\sigma^{2} + \varepsilon}K^{2}}{\sigma^{2}}(\bar{V}_{0} - \underline{V}_{0})}\right)^{2}} - \frac{\left(p_{0} - g - \frac{\sqrt{\varepsilon}K}{\sqrt{2m\sigma^{2} + \varepsilon}K^{2}}(\bar{V}_{0} - x)\right)e^{-\frac{\sqrt{\varepsilon}K + \sqrt{2m\sigma^{2} + \varepsilon}K^{2}}{\sigma^{2}}(\bar{V}_{0} - \underline{V}_{0})}e^{+\frac{\sqrt{\varepsilon}K - \sqrt{2m\sigma^{2} + \varepsilon}K^{2}}{\sigma^{2}}(x - \underline{V}_{0})}}{1 - e^{-\frac{2\sqrt{2m\sigma^{2} + \varepsilon}K^{2}}{\sigma^{2}}(\bar{V}_{0} - \underline{V}_{0})}e^{+\frac{\sqrt{\varepsilon}K - \sqrt{2m\sigma^{2} + \varepsilon}K^{2}}{\sigma^{2}}(x - \underline{V}_{0})}}e^{-\frac{\sqrt{\varepsilon}K + \sqrt{2m\sigma^{2} + \varepsilon}K^{2}}{\sigma^{2}}(\bar{V}_{0} - \underline{V}_{0})}e^{+\frac{\sqrt{\varepsilon}K - \sqrt{2m\sigma^{2} + \varepsilon}K^{2}}{\sigma^{2}}(x - \underline{V}_{0})}}\left(1 - e^{-\frac{2\sqrt{2m\sigma^{2} + \varepsilon}K^{2}}{\sigma^{2}}(\bar{V}_{0} - \underline{V}_{0})}e^{+\frac{\sqrt{\varepsilon}K - \sqrt{2m\sigma^{2} + \varepsilon}K^{2}}{\sigma^{2}}(x - \underline{V}_{0})}}\right)^{2}\right)$$

$$(50)$$

if m > 0 or $K \neq 0$, and $\mathcal{V}^S(x; p_0, K) = (p_0 - g) \left(\frac{x - \underline{V}_0}{V_0 - \underline{V}_0}\right)$ if m = 0, K = 0. On the other hand, if $x \leq \underline{V}_0$ then $\mathcal{V}^S(x; p_0, K) = 0$, and if $x \geq V_0$ then $\mathcal{V}^s(x; p_0, K) = p_0 - g$.

Proof. We shall only cover the non-trivial case where $x \in (\underline{V}_0, \overline{V}_0)$. First, let's assume that either m > 0 or $K \neq 0$. We compute $\mathcal{V}^S(x; p_0, K)$ by substituting (49) into (29):

$$\begin{split} \mathcal{V}^{S}(x;p_{0},K) &= -\frac{\sigma^{2}}{2} \int_{0}^{\infty} e^{-ms}(p_{s}-g)\partial_{v}U(s,\bar{V}_{s})ds \\ &= \sum_{k=-\infty}^{+\infty} \left((2k+1)(\bar{V}_{0}-\underline{V}_{0}) - (x-\underline{V}_{0}) \right) e^{\frac{2\sqrt{\varepsilon}Kk}{\sigma^{2}}(\bar{V}_{0}-\underline{V}_{0})} \\ &\times \int_{0}^{+\infty} \frac{(p_{0}+\sqrt{\varepsilon}Ks-g)}{\sigma\sqrt{2\pi s^{3}}} e^{-ms-\frac{((2k+1)(\bar{V}_{0}-\underline{V}_{0})-(x-\underline{V}_{0})+\sqrt{\varepsilon}Ks)^{2}}{2s\sigma^{2}}} ds \\ &= \sum_{k=0}^{+\infty} \left(p_{0}-g + \frac{\sqrt{\varepsilon}K}{\sqrt{2m\sigma^{2}+\varepsilon K^{2}}} \left((2k+1)(\bar{V}_{0}-\underline{V}_{0}) - (x-\underline{V}_{0}) \right) \right) \right) \\ &\times \exp\left(+ \frac{\sqrt{\varepsilon}K}{\sigma^{2}} \cdot 2k(\bar{V}_{0}-\underline{V}_{0}) - \frac{\sqrt{\varepsilon}K + \sqrt{2m\sigma^{2}+\varepsilon K^{2}}}{\sigma^{2}} \left((2k+1)(\bar{V}_{0}-\underline{V}_{0}) - (x-\underline{V}_{0}) \right) \right) \right) \\ &- \sum_{k=1}^{+\infty} \left(p_{0}-g + \frac{\sqrt{\varepsilon}K}{\sqrt{2m\sigma^{2}+\varepsilon K^{2}}} \left((2k-1)(\bar{V}_{0}-\underline{V}_{0}) + (x-\underline{V}_{0}) \right) \right) \right) \\ &\times \exp\left(- \frac{\sqrt{\varepsilon}K}{\sigma^{2}} \cdot 2k(\bar{V}_{0}-\underline{V}_{0}) + \frac{\sqrt{\varepsilon}K - \sqrt{2m\sigma^{2}+\varepsilon K^{2}}}{\sigma^{2}} \left((2k-1)(\bar{V}_{0}-\underline{V}_{0}) + (x-\underline{V}_{0}) \right) \right) \right) \end{split}$$

In the second equality, we switched the order of summation and integration, which can be justified by Fubini's theorem for m > 0 or $K \neq 0$. The resulting infinite series can be evaluated using standard geometric series results to yield (50). If m = 0 and K = 0, then it is known (see Branco et al. (2012)) that the seller's expected profit is $(p_0 - g) \left(\frac{x - V_0}{V_0 - V_0}\right)$. \Box

In the limit $\underline{V}_0 \to -\infty$ (i.e. the limit $c \to 0$) (50) reduces to (47) we previously studied. Unlike in the single boundary case, in the presence of the exit boundary, the expected seller's profit is not only continuous at K = 0, but also differentiable, even when m = 0, as we will show below. We now focus on the m = 0 case.

From (50) we have that $\mathcal{V}^S(x; p_0, K < 0)|_{m=0}$ is given by (30), and that:

$$\mathcal{V}^{S}(x;p_{0},K>0)|_{m=0} = \frac{(p_{0}-g-(\bar{V}_{0}+x-2\underline{V}_{0}))\exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^{2}}(\bar{V}_{0}-x)\right)}{1-\exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^{2}}(\bar{V}_{0}-\underline{V}_{0})\right)} + \frac{2(\bar{V}_{0}-\underline{V}_{0})\exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^{2}}(\bar{V}_{0}-x)\right)}{\left(1-\exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^{2}}(\bar{V}_{0}-\underline{V}_{0})\right)\right)^{2}} - \frac{(p_{0}-g-(\bar{V}_{0}-x))\exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^{2}}(\bar{V}_{0}-\underline{V}_{0})\right)}{1-\exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^{2}}(\bar{V}_{0}-\underline{V}_{0})\right)} - \frac{2(\bar{V}_{0}-\underline{V}_{0})\exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^{2}}(\bar{V}_{0}-\underline{V}_{0})\right)}{\left(1-\exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^{2}}(\bar{V}_{0}-\underline{V}_{0})\right)\right)^{2}}. \quad (51)$$

Both (30) and (51) are valid expressions for all $K \neq 0$, and with some works, we can show them to be equal for all $K \neq 0$. This proves $\mathcal{V}^S(x; p_0, K)$ is given by (30) for all $K \neq 0$. \Box

Proof of Lemma 5. We can compute that

$$\frac{\partial p_0^*}{\partial K}(x, K=0) = \frac{1}{12r\sigma} \left(3\sigma - 3\sqrt{\frac{2c^2}{r} + \sigma^2} \sinh^{-1}\sqrt{\frac{r\sigma^2}{2c^2}} - \sigma \left(\sinh^{-1}\sqrt{\frac{r\sigma^2}{2c^2}} \right)^2 \right) \le 0$$

where the inequality is strict everywhere except at $\sqrt{r\sigma}/c = 0$. Given any q, r, c, σ^2 such that $\sqrt{r\sigma} > 0$ we can find sufficiently small $\lambda_1 > 0$ such that $p_0^*(x, .)$ is a decreasing function for $K \in [-\lambda_1, +\lambda_1]$. If necessary, we can always restrict $\lambda_1 > 0$ to be smaller (i.e. $\lambda_1 < 1$) to ensure that the buyer's respond ε -optimality to $K \in [-\lambda_1, +\lambda_1]$ according to Proposition 4. Any local maximum point of $\mathcal{V}^S(x; ., .)$ would take the form $(p_0^*(x; K^*), K^*)$ where $K^* := \arg \max_K \mathcal{V}^S(x; p_0^*(x, K), K)$, and $p_0^*(x; 0) = \hat{p}(x)$, hence $p_0^* < \hat{p}, K^* \gtrsim 0$, or $p_0^* > \hat{p}, K^* \lesssim 0$.

The existence of parameters q, r, c, σ^2 which give examples of maximum point (p_0^*, K^*) satisfying $p_0^* < \hat{p}, K^* \gtrsim 0$, or $p_0^* > \hat{p}, K^* \lesssim 0$ can be found from Figure 4. In particular, at the boundary between region III and VI (or the boundary between region II and III), $(p_0^*, K^*) = (\hat{p}, 0)$ is a local maximum, and a unique one if $\lambda_1 > 0$ is sufficiently small. Due to the continuity of $\mathcal{V}^S(.; ., .)$ and $p_0^*(., .)$, by choosing a slightly higher q, we obtain an example of a maximum point with $p_0^* > \hat{p}, K^* \leq 0$, while choosing a slightly lower q, we obtain an example of a maximum point with $p_0^* < \hat{p}, K^* \gtrsim 0$.

Proof of Proposition 8. We first introduce an intuitive technical result.

Lemma 6. Consider the parameters m, r, c, σ^2 , and $\varepsilon > 0$, such that $\underline{V} > -\infty$ (i.e. c > 0). There exists $\bar{x} = \bar{x}(m, r, c, \sigma^2) > 0$ such that for all $x \ge g + \bar{x}$, there exists an ε -equilibrium: $\{\{\tau^*[p] \in \mathcal{T}\}_{p \in \mathcal{P}_T}, p^* \in \mathcal{P}_T\}$ such that $\underline{V}_t[p^*] \ge g$, for all $t \in [0, \infty)$.

Proof. Suppose by contrary that such $\bar{x} > 0$ does not exist. Given any $x \in \mathbb{R}$, we can find $p^* \in \mathcal{P}_T$ such that $\mathcal{V}^S(x; \tau^*[p^*], p^*) \geq V^S(x) - \varepsilon$, where $\tau^*[p] \in \mathcal{T}$ satisfies $\mathcal{V}^B(t, x; \tau^*[p], p) \geq V^B(t, x) - \varepsilon$, and let's denote the corresponding exit boundary by $\underline{V}_t[p^*]$. By assumption, we have $\underline{V}_t[p^*] < g$ for some $t \in [0, T]$. But since $\underline{V}[p^*]$ is continuous, for a sufficiently large $\Delta > 0$ we must have $\underline{V}_t[p^*] + \Delta > g$ for all $t \in [0, T]$. But then by assumption, $p^* + \Delta \in \mathcal{P}$ cannot be an ε -optimal pricing strategy of the seller when the buyer has an initial valuation $x + \Delta$ because $\underline{V}_t[p^* + \Delta] = \underline{V}_t[p^*] + \Delta > g$ for all t, so there must exist $p^{**} \in \mathcal{P}$ such that $\mathcal{V}^S(x + \Delta; \tau^*[p^*], p^{**}) > \mathcal{V}^S(x + \Delta; \tau^*[p^*], p^* + \Delta)$. But since $\mathcal{V}^S(x + \Delta; \tau^*[p^*], p^* + \Delta) = \mathcal{V}^S(x; \tau^*[p^*], p^*) + \Delta$, we have that

$$\mathcal{V}^{S}(x;\tau^{*}[p^{**}],p^{**}-\Delta) = \mathcal{V}^{S}(x+\Delta;\tau^{*}[p^{**}],p^{**}) - \Delta$$

> $\mathcal{V}^{S}(x+\Delta;\tau^{*}[p^{*}],p^{*}+\Delta) - \Delta = \mathcal{V}^{S}(x;\tau^{*}[p^{*}],p^{*}) \ge V^{S}(x) - \varepsilon.$

Since $\varepsilon > 0$ is arbitrary small, we have $\mathcal{V}^S(x; \tau^*[p^{**}], p^{**} - \Delta) \ge V^S(x)$. Relabelling $p^{**} - \Delta$ as p^* and repeating the argument above again we may argue that the last inequality is strict, thus establishing a contradiction. In particular, $p^{**} = p^* + \Delta'$ must be an ε -optimal pricing strategy for the seller, satisfying $\underline{V}_t[p^{**}] > g$ for all $t \in [0, \infty)$, when the buyer has an initial valuation $x + \Delta' > \underline{V}_0[p^*] + \Delta' > g$ for all $\Delta' > \Delta$, and we may define $\bar{x} := x + \Delta - g$. \Box

Equation (32) in the proposition follows from Proposition 2 and Proposition 1. Since we know that as K increases, the corresponding purchase and exit boundaries $\bar{V}_t[p_0 + Kh]$ and $\underline{V}_t[p_0 + Kh]$ will monotonically decrease and increase toward $p_0 + Kh_t$, respectively. If $x > p_0$ then only the purchase boundary $\bar{V}_0[p_0 + Kh]$ will reach x as $K \to \infty$, giving the seller the payoff $p_0 - g$. Likewise, for $x \le p_0$ only $\underline{V}_0[p_0 + Kh]$ will reach x as $K \to \infty$ giving the seller the payoff 0.

Let's define $\bar{x} := \bar{x}(m, r, c, \sigma^2)$ as in Lemma 6, then if $x \ge g + \bar{x}$ we can find an ε -optimal strategy $p \in \mathcal{P}_T$ satisfying $\underline{V}_t[p] > g$ for all $t \in [0, \infty)$. It follows that

$$\mathcal{V}^{S}(x;\tau^{*}[p],p) = \mathbb{E}\left[e^{-m\tau^{*}[p]}(p_{\tau^{*}[p]}-g)\cdot 1_{v_{\tau^{*}[p]}\geq p_{\tau^{*}[p]}} \mid v_{0}=x\right]$$

$$\leq \mathbb{E}\left[\left(p_{\tau^{*}[p]} - g\right) \cdot 1_{v_{\tau^{*}[p]} \ge p_{\tau^{*}[p]}} \mid v_{0} = x\right] \leq \mathbb{E}\left[v_{\tau^{*}[p]} - g \mid v_{0} = x\right] = x - g.$$
(52)

The first inequality followed from removing the discounting factor. The second inequality followed by noting that if v_t hits the purchase boundary $\bar{V}_t[p]$ first we would have $v_{\tau^*[p]} - g \ge p_{\tau^*[p]} - g$, and if v_t hits the exit boundary first we would have $v_{\tau^*[p]} < p_{\tau^*[p]}$, so $v_{\tau^*[p]} - g \ge 0 = (p_{\tau^*[p]} - g) \cdot 1_{v_{\tau^*[p]} \ge p_{\tau^*[p]}}$. The final equality followed from the Martingale stopping theorem since $|v_{t \land \tau^*[p]}|$ is bounded by $\max_{s \in [0,\infty)} \{|\underline{V}_s[p]|, |\bar{V}_s[p]|\} = \max_{s \in [0,T]} \{|\underline{V}_s[p]|, |\bar{V}_s[p]|\}$ where the latter is finite because both boundaries are continuous over [0,T] and are constant over $[T,\infty)$ by the definition of \mathcal{P}_T . So $x - g \ge V^S(x) - \varepsilon$ for any arbitrary $\varepsilon > 0$, hence we conclude that $V^S(x) = x - g$. The claim that this supremum can be approached by (33) follows from (32).

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