Consumer Gradual Learning and Firm Non-stationary Pricing

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March 2024

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Abstract

Consumers gather information gradually to help themselves make the purchasing decision. The increasingly popular privacy regulations have made it harder for firms to track individuals in real time. Even if a firm can track consumers' browsing behavior, it is difficult for the firm to infer whether consumers like the information they see. Without the ability to track consumer's valuation evolution about the product, one may think that firms can only offer a constant price. The major innovation of our paper is to allow the price to be a function of time rather than the consumer's current valuation of the product. We find that constant price is not always optimal for the firm. It can benefit from using non-stationary pricing strategies. When the search cost is zero, the optimal price is arbitrarily close to a constant price if the firm is perfectly patient. In contrast, the slope of the optimal price is bounded from zero if the firm discounts the future. When there is friction in search, the optimal price is non-stationary, even if the firm is perfectly patient. In particular, the firm always increases the price over time if the information is too noisy or the search cost is too high. In other cases where consumers have a stronger incentive to search, the firm charges an increasing price for consumers with high or low initial valuation, whereas charges a decreasing price for medium–value consumers.

1 Introduction

It has been widely documented that consumers gather information gradually to help themselves make the purchasing decision. They can visit the seller's website to see the product description, check reviews on the retailer's storefront, or search review articles through search engines. All these search activities help them reduce their uncertainty about the product's value, but only partially. Since the seminal paper by Weitzman (1979), many papers have studied the optimal search strategy when there are multiple alternatives or multiple attributes of a product (Weitzman, 1979; Wolinsky, 1986; Moscarini and Smith, 2001; Branco et al., 2012, 2016; Ke et al., 2016; Liu and Dukes, 2016; Ke and Villas-Boas, 2019; Ke and Lin, 2020; Guo, 2021; Yao, 2023c; Chaimanowong et al., 2023). Recent papers have started to look at the marketing implications of consumer's gradual learning activities, including information provision policies (Branco et al., 2016; Jerath and Ren, 2021; Ke et al., 2023; Yao, 2023a), search costs manipulation (Bar-Isaac et al., 2010; Dukes and Liu, 2016), product line design (Villas-Boas, 2009; Kuksov and Villas-Boas, 2010; Guo and Zhang, 2012; Liu and Dukes, 2013; Zia and Kuksov, 2023), consumers' repeat buying and drop–out decision Chaimanowong and Ke (2022), and advertising (Mayzlin and Shin, 2011). Among the possible marketing mixes, pricing is one of the most salient and important marketing strategies. Existing studies mainly consider either constant price (Branco et al., 2012) or price that depends on the current valuation of the consumers (Ning, 2021). In recent years, the increasingly popular privacy regulations such as GDPR and CCPA have made it harder for firms to track individuals in real time. Even if a firm can track consumers' browsing behavior, it is unclear from its perspective how consumers will interpret the information they see. For example, Tesla may be able to observe that a consumer clicks on an image of the interior design of the car but may not know whether the consumer likes the large screen on Tesla or the traditional dashboard. This calls into question whether the firm can track the consumer's belief evolution process when the consumer is searching for information. Though empirical papers have estimated consumer valuation with consumer search (Kim et al., 2010; Seiler, 2013; Koulayev, 2014; Bronnenberg et al., 2016; Kim et al., 2017), they also use the choice data for the estimation. In order to adjust the price during consumer search, firms cannot rely on the eventual choice decision. Furthermore, the empirical literature estimates a single value for a product per consumer. However, the consumer's valuation evolves during gradual learning. There is not enough variation to estimate such dynamic valuation evolution.

Given the difficulty mentioned above, one may think that firms can only offer a constant price when they cannot track consumers. The last hope for firms is one tool they are equipped with that can never be banned by regulations - time. The major innovation of our paper is to allow the price to be a function of time rather than the consumer's current valuation of the product. In other words, we explore non-stationary pricing strategies and ask two questions. First, is constant price always optimal for the firm when it cannot track the consumer's belief evolution process? Second, what should the firm do if the constant price is not optimal?

We find that constant price is not always optimal for the firm. It can benefit from using non-stationary pricing strategies. We prove that a consumer can do almost as well by approximating any price which is sufficiently slow–moving by linear price if she is sufficiently myopic, which can be a building block for future research to simplify the strategy space of non-Markov problems. Given this result, by assuming that the consumer is sufficiently myopic and the price is linear and varies slowly, we show that, when the search cost is zero, the optimal price is arbitrarily close to a constant price if the firm is perfectly patient. In contrast, the slope of the optimal price is bounded from zero if the firm discounts the future. When there is friction in search (positive search costs), the optimal price is non-stationary, even if the firm is perfectly patient. In particular, the firm always increases the price over time if the information is too noisy or the search cost is too high. In other cases where consumers have a stronger incentive to search, the firm charges an increasing price for consumers with high or low initial valuation, whereas charges a decreasing price for medium–value consumers.

Our contribution is twofold. On the one hand, our paper provides new managerial insights for the firm by considering non-stationary pricing. The primary goal of marketing is to reduce the cost and increase the return. Using time as the information source to guide pricing decisions is essentially free. Firms do not need to invest heavily in the tracking technology. Hence, all the increased revenue due to non-stationary pricing becomes profit. It is also immune to privacy regulations. Apple's iPhone privacy upgrades cost publishers like Facebook, YouTube, Twitter, and Snap nearly 10 billion in ad revenue in 2021 alone because the increased privacy restriction limited advertisers' ability to target consumers.[1] Privacy regulations can prevent firms from tracking consumers' demographic information, browsing behavior, and other characteristics, but cannot ban the time to which everyone has access.

Extant research mainly focuses on the economic impact of privacy regulations (Goldfarb and Tucker, 2011; Conitzer et al., 2012; Campbell et al., 2015; Athey et al., 2017; Goldberg et al., 2019; Montes et al., 2019; Choi et al., 2020, 2023; Rafieian and Yoganarasimhan, 2021; Choi et al., 2022; Baik and Larson, 2023; Ke and Sudhir, 2023; Ning et al., 2023; Yao, 2023b). We contribute to this stream of literature by studying what firms can do to respond to such privacy regulations. Not much attention has been paid to this direction. A notable exception is Bondi et al. (2023), where they study in a different context how media firms can use content design to aid their inference of consumer type. The underlying mechanism in that paper is consumer self-selection, whereas our mechanism relies on consumers' forward-looking behavior.

On the other hand, our non-stationary framework and solution method contribute theoretically to optimal control. The vast majority of papers in marketing and economics restrict attention to Markov strategies. The most common reason is tractability rather than managerial justifications. Therefore, this restriction may not be without loss of generality and may cost firms "free dollars" as shown in this paper. Marinovic et al. (2018) contrasts Markov

[1] Source: https://www.businessinsider.com/apple-iphone-privacy-facebook-youtube-twitter-snap-lose-10-billion-2021-11.

equilibria and non-Markov equilibria in a reputation model. The comparison emphasizes that restricting attention to Markov equilibria may lead to qualitatively different and unrealistic predictions, which highlights the importance of considering non-Markov strategies. To the best of our knowledge, no paper has studied the non-stationary pricing problem under consumer gradual learning. We view this paper as an important first step in understanding firms' non-Markov interventions in the presence of consumer search.

2 The Model

Our model builds on the seminal paper on consumer search, Branco et al. (2012). A firm offers a product with a marginal cost of g and chooses the price. A consumer decides whether to purchase it or not. The consumer's initial valuation is v_0, which is common knowledge. Before making a decision, she can gradually learn about various product attributes to update her belief about the product's value.

The total utility the consumer gets from consuming the product is the sum of the value of M attributes the product has. Before searching, the consumer's initial valuation for the product, v_0, is her expected utility. The initial valuation represents the consumer's knowledge about the product based on past experiences, word of mouth, or advertising. The consumer can incur a search cost c to learn more about one of the product attributes. After learning about each attribute, the consumer updates the valuation of the product by incorporating the difference between the realized utility of the searched attribute and the expected utility. Denote this difference for attribute i by x_i. Then, the consumer's valuation after searching for t attributes is $v_t := v_0 + \sum_{i=1}^{t} x_i$. Suppose that the value of the difference is binary, $x_i = \pm z$ with equal probability. When there are infinitely many attributes, each with a very small weight in value, v_t becomes a Brownian motion.

$$dv_t = \sigma dW_t,$$

where $\sigma = z/\sqrt{dt}$, the flow search cost is cdt per dt time, and $\{W_t\}_{t\in\mathbb{R}_{\geq 0}}$ is the standard Brownian motion adapted to some filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in\mathbb{R}_{\geq 0}}, \mathbb{P})$. Figure 1 illustrates the evolution of the consumer's valuation as the consumer checks more and more attributes.

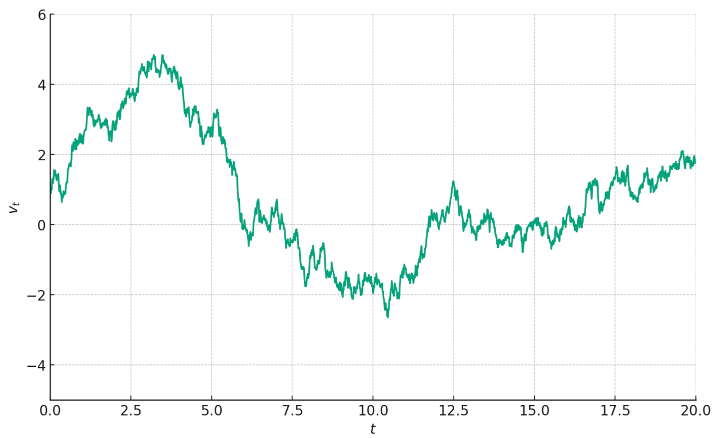

Figure 1: A sample path of the consumer's valuation evolving processes during search

Previous works that study the firm/seller's marketing strategy in the presence of consumer gradual learning either assume that the firm perfectly observes the consumer's valuation evolution processes (v_t is common knowledge between the consumer and the firm) and can condition the price on the consumer's current valuation (Ning, 2021), or that the firm charges a constant price over time (Branco et al., 2012). Suppose we view the consumer's valuation as the state variable, as is the standard and natural way of defining the state variable in the literature. The firm's strategy in the first scenario is allowed to be a function of the state variable v_t. In this case, the firm's problem is to choose the optimal *Markov strategy*. This setup does not fit all real-world examples. Due to the increasingly common privacy regulations such as GDPR and CCPA, it is harder for firms to track consumers' search behavior online. Even if some firms can track consumer's search path, it is hard to know whether or not consumers like the information they find. Moreover, in many offline settings, individual-level tracking is not feasible.

When the firm cannot observe the consumer's valuation evolution and therefore cannot choose the price based on v_t, is offering a constant price the best it can do? The major innovation of this paper is to allow the price to be a function of time t rather than the state variable v_t. In other words, the firm can strategically explore *non-stationary* pricing strategies. Formally, the firm can commit to a pricing scheme $p := \{p_t\}_{t\geq 0} \in \mathcal{P}$, where $\mathcal{P}$ is a subset of continuously differentiable functions on $[0,\infty)$, $\mathcal{P} \subset C^1[0,\infty)$. This pricing strategy is a *non-Markov strategy* because p_t depends on history (time t) other than the current state v_t. It is widely known in optimal control that it is much harder to characterize *non-Markov strategies* than *Markov strategies*.

The consumer search strategy consists of choosing an appropriate stopping time and we denote by $\mathcal{T}$ the set of all stopping times adapted to $\{\mathscr{F}_t\}_{t\in\mathbb{R}_{\geq 0}}$. We formalize the setup as a game with two players, a consumer ("Buyer" B) and a firm ("Seller" S), playing in the following sequence:

1. At $t = 0$, the firm commits to a pricing strategy $p \in \mathcal{P} \subset C^1[0,\infty)$.
2. At any $t > 0$, the consumer decides whether to purchase the product, exit, or search for more information.
3. The game ends when the consumer makes a purchase or exits.

The only knowledge the seller has about the consumer is their initial valuation, v_0, which may be derived from a survey conducted over a large population. Importantly, when the consumer decides whether to purchase the product, exit, or keep searching at any given time, she takes into account both the current price and the future price trajectory. For any $p \in \mathcal{P}$ and $\tau \in \mathcal{T}$, we define [2]

$$\mathcal{V}^B(t,x;\tau,p) := \mathbb{E}\left[e^{-r(\tau-t)}\max\{v_\tau - p_\tau, 0\} - \int_t^\tau ce^{-r(s-t)}ds \mid v_t = x\right] \tag{1}$$

[2] For simplicity, we use p to denote $\{p_t\}_{t\geq 0}$ whenever this does not cause confusion.

and

$$\mathcal{V}^S(x;\tau,p) := \mathbb{E}\left[e^{-m\tau}(p_\tau - g)\cdot 1_{v_\tau \geq p_\tau} \mid v_0 = x\right] \tag{2}$$

as the corresponding consumer's, and the firm's expected payoffs, respectively.

Commitment Assumption

We assume that the firm has dynamic commitment power. It would be a relatively strong assumption if the firm could track consumer valuation processes due to the hold-up problem. When new information (consumer's current valuation) arrives, the firm has an incentive to deviate from the pricing scheme announced at the beginning to extract more surplus from the consumer. Anticipating the firm's incentive to deviate, the consumer will not start searching without a commitment device. Ning (2021) shows that a firm without commitment power can address the hold-up problem by offering a "list price." In contrast, the firm does not need to offer a list price if it has dynamic commitment power. In such cases, whether the firm has commitment power will lead to qualitatively different results.

In our setting, the firm does not receive new information over time (no updates on consumer valuation v_t). Therefore, there is no hold-up problem and the firm does not have an incentive to deviate from the announced pricing strategy. Thus, the commitment assumption is not a strong assumption in this case. We make the assumption mainly for a cleaner analysis and presentation.

Solution Concept

We are interested in the following equilibrium concept.

Definition 1. *An ε–Subgame perfect Nash's equilibrium (ε–SPNE) consists of:*

$$(\{\tau^*[p] \in \mathcal{T}\}_{p\in\mathcal{P}}, p^* \in \mathcal{P})$$

such that: for all $p \in \mathcal{P}$,

$$\mathcal{V}^B(t,x;\tau^*[p],p) \geq \mathcal{V}^B(t,x;\tau,p) - \varepsilon, \quad \forall \tau \in \mathcal{T},$$
$$\textit{and} \;\; \mathcal{V}^S(x;\tau^*[p^*],p^*) \geq \mathcal{V}^S(x;\tau^*[p],p) - \varepsilon, \quad \forall p \in \mathcal{P}.$$

The consumer's value function given the seller's pricing strategy p is:

$$V^B(t,x;p) := \sup_{\tau\in\mathcal{T}} \mathcal{V}^B(t,x;\tau,p). \tag{3}$$

When there is no ambiguity, we will compactly write $V^B(t,x) = V^B(t,x;p)$. Analogously, we define the seller's value function:

$$V^S(x) := \sup_{p\in\mathcal{P}} \mathcal{V}^S(x;\tau^*[p],p) \tag{4}$$

Our choice of the equilibrium concept is motivated by the greater analytical traceability of the problem via perturbation theory to the order of ε. For instance, we later solve for an analytical closed form of a myopic consumer's strategy to a slow moving pricing using linear approximation to the order of ε. For further discussion we refer to Assumption 1. There is also a technical reason for such an equilibrium concept as we do not need to be concerned about the existence of $\tau^*[p] \in \mathcal{T}$ or $p^* \in \mathcal{P}$ that achieve the supremum. In a certain case, it is possible to show that the firm's profit supremum can only be approached via a limit of an admissible pricing strategy (we did not require $\mathcal{P}$ to be closed in general).

3 Consumer's Strategy

The consumer faces an optimal stopping problem. She needs to determine the purchasing and quitting boundaries at any time. When the price is non-stationary, the consumer's purchasing and quitting boundaries are also time-contingent. This time-varying property makes her optimal stopping problem challenging even if we fix a pricing scheme. To illustrate the impact of non-stationary pricing on the consumer's problem, we first review the benchmark with constant price, which has been characterized in Branco et al. (2012).

3.1 A Constant Price Benchmark

If the price is constant, $p_t = p_0 \in \mathbb{R}$, then intuitively, the consumer's strategy and value function do not depend on time, $V^B(t, x; p_0) = V_0^B(x; p_0), \bar{V}_t = \bar{V}$, and $\underline{V}_t = \underline{V}$. The value function of the consumer satisfies:

$$\frac{\sigma^2}{2}\partial_x^2 V_0^B - rV_0^B - c = 0$$

Instead of having a value matching and a smooth pasting condition at any time t, we now only have a value matching condition and a smooth pasting condition for the entire problem:

$$\begin{aligned} V_0^B(\bar{V}; p_0) = \bar{V} - p_0, \quad \partial_x V_0^B(\bar{V}; p_0) = 1, \\ V_0^B(\underline{V}; p_0) = 0, \quad \partial_x V_0^B(\underline{V}; p_0) = 0. \end{aligned}$$

The stationary structure leads to closed-form solutions.

$$\begin{aligned} &V_0^B(x; p_0) = \frac{c}{r}\left[\cosh\frac{\sqrt{2r}}{\sigma}(x - \underline{V} - p_0) - 1\right], \\ &\bar{V} = p_0 + \sqrt{\frac{c^2}{r^2} + \frac{\sigma^2}{2r}} - \frac{c}{r}, \\ &\underline{V} = p_0 + \left(\sqrt{\frac{c^2}{r^2} + \frac{\sigma^2}{2r}} - \frac{c}{r}\right) - \frac{\sigma}{\sqrt{2r}}\log\left(\sqrt{\frac{r\sigma^2}{2c^2}} + \sqrt{1 + \frac{r\sigma^2}{2c^2}}\right). \end{aligned}$$

Comparing this benchmark and our problem, we can see that stationarity simplifies the problem significantly. In the benchmark model, the consumer's entire optimal stopping strategy can be summarized by **two unknowns**: the purchasing threshold $\bar{V}$ and the quitting threshold $\underline{V}$, which does not depend on time t. The consumer will purchase the product at any time during the search if her valuation reaches the purchasing threshold and will quit searching at any time if her valuation reaches the quitting threshold. In contrast, the consumer's entire optimal stopping strategy consists of **an infinite number of unknowns**. Knowing that the price changes over time, the consumer's purchasing and quitting thresholds also evolve. She has different purchasing and quitting thresholds at different times. So, instead of pinning down a one-dimensional purchasing/quitting threshold, we need to determine a two-dimensional purchasing/quitting boundary. These time-dependent thresholds significantly complicate our problem.

3.2 Consumer's Strategy under Non–Stationary Pricing

Let's consider the case where the set of admissible pricing strategies $\mathcal{P}$ is given by the set of continuously differentiable functions that are constant after some amount of time $T > 0$:

$$\mathcal{P}_T := \left\{ p \in C^1[0, \infty) \mid p_t = p_T, \forall t \geq T \right\}.$$

We consider when $T > 0$ is large but finite to provide the boundary condition at $t = T$ needed for the existence and uniqueness result, but in many cases, there will be no problem taking the limit $T \to \infty$. Based on the constant price result in the previous section, and by deriving the Hamilton–Jacobi–Bellman (HJB) equation corresponding to the optimization problem (3), we propose to consider the following free–boundary problem given $p \in \mathcal{P}_T$:

Find $V : [0, T] \times \mathbb{R} \to \mathbb{R}$, with $V(t, .)$ convex, and continuous functions $\bar{V}[p], \underline{V}[p] : [0, T] \to \mathbb{R}$, with $\bar{V}_t[p] \geq \underline{V}_t[p]$, such that

$$
\begin{cases}
\frac{\sigma^2}{2}\partial_x^2 V + \partial_t V - rV - c = 0, & (t,x) \in \Omega \\
V(t, \bar{V}_t[p]) = \bar{V}_t[p] - p_t, & \partial_x V(t, \bar{V}_t[p]) = 1, \\
V(t, \underline{V}_t[p]) = 0, & \partial_x V(t, \underline{V}_t[p]) = 0, \\
V(T, x) = V_0^B(x; p_T),
\end{cases} \tag{5}
$$

where

$$
\Omega := \{(t,x) \in [0,T] \times \mathbb{R} \mid \underline{V}_t[p] < x < \bar{V}_t[p]\}.
$$

By restricting p to be constant p_T for all $t \geq T$, we automatically have that $V^B(T, x; p)$ must be given by the constant price value function $V_0^B(x; p_T)$, which makes sense of the 'initial' condition. The non–stationary analog of the usual value–matching and smooth–pasting conditions are also intuitive. The decision boundaries are the constant price p_T boundaries for $t \geq T$ so we may view $\bar{V}[p]$, $\underline{V}[p]$ as functions $[0, \infty) \to \mathbb{R}$ by extending their definition by this constant. When it is clear from the context, we will write $\bar{V}_t[p], \underline{V}_t[p]$ simply as $\bar{V}_t, \underline{V}_t$.

To solve the consumer's problem, we first need to establish a result for the existence and uniqueness of the solution to (5). In stochastic control problems, the strong solution (solution in the usual sense) to the PDE does not always exist. The standard approach is to work with a relaxed notion, the weak solution (Bressan, 2012; Evans, 2022). Readers who are not concerned with such technical detail can skip to the end of Remark 1.

Given any open $G \subset \mathbb{R}^n$ and $1 \leq p \leq \infty$, we recall the standard notations in functional analysis $L^p(G)$ and $L^p_{loc}(G)$ are standard in functional analysis and we refer to Bressan (2012); Evans (2022) for more detail. Given a Banach space X, we also borrow the notation $L^p(0, T; X)$ from §5.9.2 Evans (2022) to denote the space of all $u : [0, T] \to X$ such that $\left(\int_0^T \|u(t)\|_X^p dt\right)^{1/p} < \infty$ if $1 \leq p < \infty$, or $\operatorname{ess\,sup}_{t \in [0,T]} \|u(t)\|_X < \infty$ if $p = \infty$. Similarly, the notation $C(0, T; X)$ denotes the space of continuous functions $u : [0, T] \to X$.

Definition 2. *(Weak derivative) Let $G \subset \mathbb{R}^n$ be open. For any $f \in L^p(G)$ and $\alpha \in \mathbb{Z}^n_{\geq 0}$ the α-weak derivative of f (if exists) is the function $v \in L^p(G)$ such that for all compactly* *supported smooth functions $\phi \in C_c^\infty(G)$ we have*

$$\int_G f(x)\partial^\alpha \phi(x)dx = (-1)^{|\alpha|} \int_G g(x)\phi(x)dx,$$

where $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$. If such v exists, it is a.e. unique, and we write $\partial^\alpha f := v$. We denote by $W^{q,p}(G) \subset L^p(G)$ the set of all f such that all α-weak derivatives with $|\alpha| \leq q$ exists, and by $W_{loc}^{q,p}(G) \subset L_{loc}^p(G)$ the set of all f such that $f|_{G'} \in W^{q,p}(G')$ for any open G' compactly contained in G.

(Weak solution) A weak solution to the free–boundary problem (5) on Ω is any $V \in L^2(0, T; W_{loc}^{2,2}(\mathbb{R}))$ with a weak derivative $\partial_t V \in L^2(0, T; W_{loc}^{1,2}(\mathbb{R})^*)$ such that

$$\int_\Omega \left(\frac{\sigma^2}{2} \partial_x^2 V(t, x) + \partial_t V(t, x) - rV(t, x) - c \right) \phi(t, x)dx = 0$$

for all $\phi \in C_c^\infty(\Omega)$.

Now, we can state the existence and uniqueness result.

Lemma 1. *Given a pricing strategy $p \in \mathcal{P}_T$, we have the following.*

1. *(Existence) The value function $V^B(., .; p)$ is monotonically increasing, convex in x, and gives a weak solution to the free boundary problem (5) with $V^B(., .; p) \in L^\infty(0, T; W_{loc}^{2,\infty}(\mathbb{R}))$, and weak derivative $\partial_t V^B(., .; p) \in L^\infty(0, T; L_{loc}^\infty(\mathbb{R}))$.*

2. *(Uniqueness) If V is a weak solution to the free boundary problem (5) with $V \in L^\infty(0, T; W_{loc}^{2,\infty}(\mathbb{R}))$ and weak derivative $\partial_t V \in L^\infty(0, T; L_{loc}^\infty(\mathbb{R}))$ then $V = V^B(., .; p)|_\Omega$.*

Remark 1. *By asking for a weak solution $V(t, .) \in L^2(0, T; W_{loc}^{2,2}(\mathbb{R}))$ we guarantee that $\partial_x V(t, .)$ exists in the classical sense, hence the boundary conditions for $\partial_x V(t, .)$ make sense. In fact, the solution discussed in Lemma 1 is reasonably nice. According to §5.9.2. Evans (2022), the solution can be represented by $V \in C(0, T; W_{loc}^{2,\infty}(\mathbb{R}))$, and since $V(t, .)$ is convex in x it follows that V is continuous, and it is then automatic that $\bar{V}[p], \underline{V}[p]$ are continuous.*

Further, $\partial_t V(.,x)$ and $\partial_x^2 V(t,.)$ coincide a.e. with some bounded functions (not necessarily continuous), and the PDE in (5) is satisfied on Ω. Therefore, in our context, nothing much is lost in imagining $\partial_t V$ and $\partial_x^2 V$ as classical derivatives, which is also why we are using the symbol ∂^α for both weak and classical derivatives as opposed to D^α in most functional analysis literature.

The lemma above implies that we can work with (5) to solve for the consumer's value function instead of directly finding the optimal $\tau^*[p] \in \mathcal{T}$. A weak solution is a sufficiently well–behaved concept for our work as we will mostly be interested in the consumer's decision boundaries rather than the smoothness properties of the value function itself. More importantly, the value function (hence the decision boundaries) does not change by much corresponding to any small changes in the given pricing strategy.

Lemma 2. *Let $p, q \in \mathcal{P}_T$ and let $V(.,.;p)$ and $V(.,.;q)$ be the corresponding solution to the free–boundary problem (5), then $|V(t,x;p) - V(t,x;q)| \leq \max_{s\in[t,T]} e^{-r(s-t)}|p_s - q_s|$ for all $(t,x) \in [0,T] \times \mathbb{R}$.*

Remark 2. *Lemma 2 shows that for discounting consumers $r > 0$, any changes in price far in the future do not have much effect in the present. We can make sense of the consumer response to an arbitrary $p \in C^1[0,\infty)$ such that $\lim_{t\to\infty} e^{-rt} p_t = 0$. We find the solution $V(.,.;p^T)$ to $p^T \in \mathcal{P}_T$ according to Lemma 1 where p^T is given by p over $[0, T-\varepsilon]$ and constant for all $t \geq T$, then for all sufficiently large T we have $\left|V\left(t,x;p^{T'}\right) - V\left(t,x;p^{T''}\right)\right| < \varepsilon$ for all $T', T'' > T$. This is a real–valued Cauchy sequence, so we may define the value function of an infinite horizon p to be the limit if we wish.*

Lemma 2 justifies our perturbative studies of the solution. To be consistent with ε–equilibrium concept, we consider perturbations of $\sqrt{\varepsilon}$-order. In particular, an $\sqrt{\varepsilon}$-order changes in p will results in $\sqrt{\varepsilon}$-order changes in the value of V, and the boundaries $\bar{V}[p], \underline{V}[p]$. The following applies this idea to investigate the direction of consumers' reactions to some small changes in pricing, giving us a better idea of the structure of the solution.

Proposition 1. *Let $p \in \mathcal{P}_T$ be a pricing strategy, and let $\bar{V}[p], \underline{V}[p] : [0, \infty) \to \mathbb{R}$ denotes the corresponding purchase and exit boundaries. Let $h \in \mathcal{P}_T$ be an arbitrary pricing strategy monotonically increasing over $[0, T]$, and $K \in \mathbb{R}$ be a constant. Then under the pricing strategy $\tilde{p} := p + \sqrt{\varepsilon} K h$, the ε–optimal purchase and exit boundaries take the form*

$$\begin{aligned} \bar{V}[\tilde{p}] &= (\bar{V}[p] + \sqrt{\varepsilon} K h) + \sqrt{\varepsilon} \bar{R} + O(\varepsilon), \quad \bar{R} = K \bar{S} \\ \underline{V}[\tilde{p}] &= (\underline{V}[p] + \sqrt{\varepsilon} K h) + \sqrt{\varepsilon} \underline{R} + O(\varepsilon), \quad \underline{R} = K \underline{S} \end{aligned} \tag{6}$$

for some continuous functions $\bar{S} : [0, \infty) \to \mathbb{R}_{\leq 0}$, and $\underline{S} : [0, \infty) \to \mathbb{R}_{\geq 0}$.

Proposition 1 states that the exit boundary is higher, and the purchase boundary is lower when the consumer expects the price to increase faster. The opposite is true when the consumer expects the price to decrease faster. On the opposite end of the spectrum, it is also interesting to understand in general how the solution behave asymptotically under a large variation of pricing strategy.

Proposition 2. *Let $p \in \mathcal{P}_T$ be a pricing strategy, and let $\bar{V}[p], \underline{V}[p] : [0, \infty) \to \mathbb{R}$ denotes the corresponding purchase and exit boundaries. Let $h \in \mathcal{P}_T$ be an arbitrary pricing strategy strictly monotonically increasing over $[0, T]$, and $K \in \mathbb{R}$ be a constant. Then the purchase and exit boundaries under $\tilde{p} := p + K h$ satisfies:*

$$\begin{aligned} \bar{V}_t[\tilde{p}] - \underline{V}_t[\tilde{p}] &\to 0, \quad \text{as } K \to +\infty \\ \bar{V}_t[\tilde{p}] - \underline{V}_t[\tilde{p}] &\to +\infty, \quad \text{as } K \to -\infty \end{aligned}$$

at any $t \in [0, T)$. In particular, $\bar{V}_t[\tilde{p}] \to \tilde{p}_t+, \underline{V}_t[\tilde{p}] \to \tilde{p}_t-$ as $K \to +\infty$, at any $t \in [0, T)$.

Solving (5) in full generality is beyond the scope of this research. For most of the remainder of this paper, we focus on the setting where analytically tractable solutions can be obtained, in particular when the pricing is a linear function in time. The fact that the space of linear pricing is much smaller than the general pricing space also simplifies the problem, especially when searching for the firm's optimal pricing strategy later in §4. Consideration of linear pricing may seem restrictive, but the following result shows that for myopic enough ε–optimal consumers, any pricing strategies which is sufficiently slow–moving can be approximated by linear pricing. Intuitively, unless the price changes very drastically in the far future such as growing super–exponentially, the myopic consumers do not look too far into the future, and over any sufficiently short time interval any differentiable functions *look like* a linear function.

Lemma 3. *(Almost optimality of linear price approximation) Let $p \in \mathcal{P}$ be a continuously differentiable pricing policy, with bounded slope:* $\sup_{t\in\mathbb{R}_{\geq 0}} |p'_t| \leq \lambda_1$*. Let $l : t \mapsto l_t := p_0 + p'_0 \cdot t$ be the linear approximation pricing policy. By assumption, for any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$|p_t - l_t| < \delta\varepsilon/2, \quad \forall t \in [0, \delta).$$

If r is sufficiently large, for instance, such that:

$$e^{-r\delta}\left(2\lambda_1 T + \bar{\bar{V}} + \frac{c}{r}\right) + \frac{\lambda_1 e^{-1}}{r} < \frac{(1-\delta)\varepsilon}{4}$$

where

$$T := \frac{2}{r}\log\frac{4\lambda_1}{\varepsilon}, \quad \bar{\bar{V}} := \frac{\lambda_1 + \sqrt{\lambda_1^2 + 2r\sigma^2}}{2r}\log\left(1 + \frac{\sqrt{\lambda_1^2 + 2r\sigma^2}}{c}\right).$$

Let $\tau^*[l] \in \mathcal{T}$ be the consumer's optimal learning strategy given the linear pricing l, then $\tau^*[l]$ is also a consumer's ε–optimal stopping time under the p pricing strategy:

$$\mathcal{V}^B(t, x; \tau^*[l]; p) \geq V^B(t, x) - \varepsilon.$$

We summarize the two simplifying assumptions to obtain analytic results and provide several justifications in the following. They are by no means minimal. However, they do not substantially limit our contribution.

Simplifying Assumptions and Discussion

Assumption 1. *For a given $\varepsilon > 0$, we assume that*

- *consumers are ε–optimal, and is sufficiently myopic*
- *the firm adjusts the price slowly over time: $|p_t'| \in O(\sqrt{\varepsilon})$*

such that the conditions for Lemma 3 are satisfied.

The assumption ensures that the given price function can be approximated by a linear function based on Lemma 3 (i.e. $r >> 0$ is large, and $|p_t'| \leq \lambda_1$ is small). Given any pricing $p \in \mathcal{P}$ the consumer will derive the learning strategy based on

$$t \mapsto p_0 + \sqrt{\varepsilon} K t, \qquad \sqrt{\varepsilon} K := p_0' \in O(\sqrt{\varepsilon}). \tag{7}$$

Of course, such linear pricing does not belong to $\mathcal{P}_T$ for any $T > 0$, however, this is not a problem according to Remark 2 which perfectly applies under Assumption 1. We do not need the consumer to be completely myopic ($r = \infty$). The consumer is still forward-looking and rationally anticipates the price evolution in the future. So, the consumer still has time-varying purchasing and quitting boundaries, and we still capture the consumer's equilibrium response to the firm's non-stationary pricing. This assumption means that the consumer cares about the near future more than the far future and simplifies the determination of the optimal stopping time. Since ε can be arbitrarily small, focusing on the consumer's ε–optimal strategy rather than the optimal strategy also does not lose much. To summarize, our assumption still captures the main driving force of non-stationary pricing and the consumer's rational response. In addition, the consumer has time-varying purchasing and quitting boundaries under linear pricing. So, we still capture the consumer's equilibrium response to the firm's non-stationary pricing.

Solution

Given that the consumer will derive the learning strategy based on linear pricing (7) we can transform to a simpler frame of reference where the consumer valuation process is a drifted Brownian motion $v_t = -\sqrt{\varepsilon}Kt + \sigma W_t$ with the price fixed at p_0. The transformed problem is stationary in time, with the corresponding HJB

$$\frac{\sigma^2}{2}\partial_x^2 V(x) - \sqrt{\varepsilon}K\partial_x V(x) - rV(x) - c = 0.$$

Therefore, the free–boundary problem (5) can be solved in this case by first solving the HJB above, before making an inverse transformation back to the original frame of reference.

Proposition 3. *Under a pricing strategy $p \in \mathcal{P}$ with $\sqrt{\varepsilon}K := p_0'$, and such that Assumption 1 is satisfied, there is an ε-optimal consumer learning strategy with the value function taking the form*

$$V^B(t,x) = A_1 e^{\frac{\sqrt{\varepsilon}K - \sqrt{\varepsilon K^2 + 2r\sigma^2}}{\sigma^2}(x - p_0 - \sqrt{\varepsilon}Kt)} + A_2 e^{\frac{\sqrt{\varepsilon}K + \sqrt{\varepsilon K^2 + 2r\sigma^2}}{\sigma^2}(x - p_0 - \sqrt{\varepsilon}Kt)} - \frac{c}{r} \tag{8}$$

with purchase and exit boundaries given by

$$\bar{V}_t = p_0 + \bar{V}[\sqrt{\varepsilon}K] + \sqrt{\varepsilon}Kt, \qquad \underline{V}_t = p_0 + \underline{V}[\sqrt{\varepsilon}K] + \sqrt{\varepsilon}Kt \tag{9}$$

where the constants $\bar{V}[\sqrt{\varepsilon}K]$, $\underline{V}[\sqrt{\varepsilon}K], A_1,$ and A_2 are determined by boundary conditions in the appendix. To the ε–order, $\bar{V}[\sqrt{\varepsilon}K]$ and $\underline{V}[\sqrt{\varepsilon}K]$ take the following analytical form,

$$\bar{V}[\sqrt{\varepsilon}K] = \bar{V} + \sqrt{\varepsilon}\bar{R} + O(\varepsilon), \qquad \underline{V}[\sqrt{\varepsilon}K] = \underline{V} + \sqrt{\varepsilon}\underline{R} + O(\varepsilon), \tag{10}$$

where

$$\underline{S} := \frac{\underline{R}}{K} = \left(\frac{\bar{V} - \underline{V}}{\sigma^2}\right)\left(\bar{V} + \frac{c}{r}\right) - \frac{1}{2r} = \frac{\sqrt{\frac{c^2}{r^2} + \frac{\sigma^2}{2r}}}{\sigma\sqrt{2r}} \log\left(\sqrt{\frac{r\sigma^2}{2c^2}} + \sqrt{1 + \frac{r\sigma^2}{2c^2}}\right) - \frac{1}{2r} > 0$$

$$\bar{S} := \frac{\bar{R}}{K} = \underline{S} - \frac{1}{2r} \cdot \frac{\bar{V} - \underline{V}}{\bar{V} + c/r} = \frac{1/(\sigma\sqrt{2r})}{\sqrt{\frac{c^2}{r^2} + \frac{\sigma^2}{2r}}} \cdot \frac{c^2}{r^2} \log\left(\sqrt{\frac{r\sigma^2}{2c^2}} + \sqrt{1 + \frac{r\sigma^2}{2c^2}}\right) - \frac{1}{2r} < 0.$$

Compared to the result of Proposition 1, we have that $\bar{R}$, $\underline{R}$ are constant in this case. Compared to the constant price benchmark, an increasing pricing scheme ($K > 0$) with the same initial price has two impacts on the purchasing threshold. On the one hand, the benefit of learning becomes lower because the consumer needs to pay more in the future if she receives positive information and likes the product more. Rationally anticipating this, the consumer has a lower incentive to search and is more inclined to purchase now, reducing the purchasing threshold (captured by the negative $\sqrt{\varepsilon}K\bar{S}$ term). On the other hand, a higher price makes the consumer less willing to purchase, raising the purchasing threshold (captured by the positive $\sqrt{\varepsilon}Kt$ term). Since the first effect remains stable while the second effect increases over time, the purchasing threshold is lower than the benchmark threshold at the beginning but eventually exceeds the benchmark threshold as the price keeps increasing.

An increasing pricing scheme also has two impacts on the quitting threshold. Both a lower benefit of searching and a higher price make it more likely for the consumer to quit. So, the quitting threshold is always higher than the benchmark threshold. We also find that the consumer searches in a narrower region (smaller $\bar{V}_t - \underline{V}_t$) if the price increases rather than staying constant because of the lower benefit of searching.

A decreasing pricing scheme ($K < 0$) has the opposite impact on the purchasing and quitting thresholds. The purchasing threshold is higher than the benchmark threshold at the beginning because the consumer has a stronger incentive to search and is less inclined to purchase immediately. It eventually falls below the benchmark threshold as the price keeps decreasing. The quitting threshold is always lower than the benchmark threshold because the benefit of both searching and purchasing is higher. Also, the consumer searches in a broader region.

4 Firm's Strategy

4.1 Firm's Expected Payoff

The expected payoff for the firm implementing the pricing strategy $p \in \mathcal{P}$ with marginal cost g is given by $\mathcal{V}^S(x;\tau^*[p],p)$ as given by (2) where $\tau^*[p] \in \mathcal{T}$ denotes the consumer's ε–optimal response to p. The formula (2) is rather abstract, in this section we show how to compute $\mathcal{V}^S(x;\tau^*[p],p)$ which we shall denote by $\mathcal{V}^S(x;p)$ hereafter for simplicity. For the consumer with initial valuation x, let $\bar{V}[p], \underline{V}[p] : [0,\infty) \to \mathbb{R}$ denotes the consumer's decision boundaries corresponding to the $\tau^*[p]$ learning strategy, we solve the heat equation with absorbing boundary condition:

$$\begin{cases} \dfrac{\sigma^2}{2}\partial_v^2 U(t,v;x) - \partial_t U(t,v;x) = 0, & (t,v) \in \Omega \\ U(t,\bar{V}_t[p];x) = 0, & U(t,\underline{V}_t[p];x) = 0 \cdot \\ U(t=0,v;x) = \delta(v-x) \end{cases} \tag{11}$$

Where $\Omega := \{(t,v) \in [0,\infty) \times \mathbb{R} \mid \underline{V}_t[p] < v < \bar{V}_t[p]\}$, and $\delta(v-x)$ denotes the Dirac–Delta distribution concentrated at x. When it is clear from the context, we may denote $U(t,v;x)$ simply as $U(t,v)$. The solution $U(t,v;x)$ exists (see Rodrigo and Thamwattana (2021) for an explicit construction of the heat equation solution with moving absorbing boundaries) and we shall *assume* that it is the probability density at time t of the consumer valuation being at $v_t = v$. The probability flux of consumer hitting the moving purchase boundary, thus getting absorbed, at time s is given by

$$-\frac{\sigma^2}{2}\partial_v U(t,\bar{V}_t) - \bar{V}_t' \cdot U(t,\bar{V}_t) = -\frac{\sigma^2}{2}\partial_v U(t,\bar{V}_t)$$

where $\bar{V}_t'$ is the time weak derivative of the purchase boundary, but the term nevertheless vanishes by construction since $U(t, \bar{V}_t) = 0$. Hence, if $x \in [\underline{V}_t, \bar{V}_t]$ then we have that

$$\mathcal{V}^S(x; p) = -\frac{\sigma^2}{2} \int_0^\infty e^{-mt}(p_t - g)\partial_v U(t, \bar{V}_t)dt, \tag{12}$$

otherwise, we simply have $\mathcal{V}^S(x; p) = (p_0 - g)1_{x \geq \bar{V}_0}$, i.e. the consumer purchases immediately and the game ends at $t = 0$.

4.2 Direction of Price Evolution

The most important property of the firm's optimal pricing strategy is the direction of price evolution. Whether the price should stay constant, increase, or decrease over time? In this section, we carry–on the previous Assumption 1. Unless mentioned otherwise, we will restrict the firm to implementing only linear pricing, that is we let the set of admissible pricing to be:

$$\mathcal{P}_{lin} := \left\{ t \mapsto p_0 + \sqrt{\varepsilon} K t \mid p_0 \in \mathbb{R}, K \in [-1, +1] \right\} \subset C^1[0, \infty).$$

Given the assumptions, the consumer will respond to $p \in \mathcal{P}_{lin}$ with learning strategy as given in Proposition 3 (and Remark 2 taking care of the boundary issues). Therefore, the firm only needs to determine the optimal (p_0, K), and we denote the expected payoff by $\mathcal{V}^S(x; p_0, K)$. Considering linear pricing from the firm's perspective is not without loss of generality. Nevertheless, linear pricing suffices to answer our first main question fully. By showing that the firm can improve its expected profit by increasing or decreasing the price linearly, we know that constant price is not generally optimal for the firm when it cannot track the consumer's belief evolution process. The most important qualitative property of the firm's optimal pricing strategy is the direction of price evolution. Linear pricing is general enough for us to see whether the price should stay constant, increase, or decrease over time, which is managerially relevant to firms in their pricing decision.

The discussion of linear pricing also serves as a template for understanding pricing strategies in more general settings where $\mathcal{P}$ could include non–linear pricing strategies as long as Assumption 1 holds. With these assumptions, the consumer ε–optimal learning decision to any $p \in \mathcal{P}$ is entirely determined by the value of p_t and its slope p'_t at any given time t according to Proposition 3 which falls under our linear pricing framework. In practice, due to some regulations $\mathcal{P}$ could involve a restriction on how fast the firm can adjust the price over time. Moreover, suppose that the firm is sufficiently myopic ($m >> 0$), or if the product search is very informative ($\sigma^2 >> 0$) so that the consumer's valuation diffuses and absorbed rapidly, or $\mathcal{P}$ involves a restriction on the pricing function's second derivative. Then we can argue that any $p \in \mathcal{P}$ is approximately linear in the foreseeable future concerning the firm's ε–optimal profit. Therefore, by finding a local maximum (p_0^*, K^*) of $\mathcal{V}^S(x; ., .)$ with $K \sim 0$, given a sufficiently restrictive non–linear $\mathcal{P}$ and certain parameters settings, (p_0^*, K^*) gives an initial price and slope of the ε–optimal pricing over $\mathcal{P}$ for the firm to implement at $t = 0$. We shall return to elaborate further on this toward the end of this section. For $t > 0$ under such settings, the linear pricing payoff $\mathcal{V}^S(v; ., .)$ can be integrated over $v \sim$ distribution of the diffused valuation v_t, then computing the optimal pricing slope to continue evolving the ε–optimal strategy. Although a further analysis of the pricing dynamic based on this outline should be possible we shall leave such a challenging topic for future research.

We discuss two linear pricing cases. In the first case, the consumer has zero search costs (but still discounts the future). In the second case, the consumer has a positive search cost.

Zero Search Costs

When the consumer has zero search costs, the continuation value of searching is positive, whereas the payoff from quitting is zero. Therefore, she would never quit searching without purchasing the product. Equivalently, the consumer's quitting boundary is $-\infty$. The consumer's optimal search strategy only has a single boundary $\bar{V}_t$.

If the firm is perfectly patient, it will not have a direct incentive to speed up the consumer's decision-making process. A purchase at any time gives the firm the same payoff. Hence, it does not have a strong incentive to increase the price over time to push the consumer to make an early decision. In addition, the firm will charge a sufficiently high price such that the consumer's payoff from purchasing the product is negative initially. Therefore, discounting does not reduce the consumer surplus if she delays the decision by searching for more information. Even if the price does not decrease over time, the consumer will keep searching for information because she has nothing to lose. So, the firm has no incentive to reduce the price over time to prevent the consumer from quitting. In sum, in this case, the firm has little incentive to charge non-stationary prices. The following proposition shows that the optimal price is arbitrarily close to a constant price when the firm is perfectly patient. On the contrary, the firm charges non-stationary prices if it discounts the future.

Proposition 4. *Suppose the search cost is zero $c = 0$.*

When the firm is perfectly patient, $m = 0$, for any fixed initial price p_0 the firm can approach the profit supremum

$$V^S(x) = \sup_{(p_0, K)} \mathcal{V}^S(x; p_0, K) = 2p_0 + \frac{\sigma}{\sqrt{2r}} - g - x,$$

and, if p_0 is not fixed, then it is optimal for the firm to set p_0 as large as possible.

When the firm's discount factor is sufficiently small or sufficiently large, the slope K of the optimal linear pricing is bounded from zero.

Intuitively, since there's no exit boundary, a consumer started at any x will eventually purchase, and $m = 0$ means the seller can wait indefinitely, therefore the seller can charge an arbitrarily high price p_0. The fact that the optimal p_0 is unbounded is also mentioned in Branco et al. (2012).

Positive Search Costs

The previous section shows $K \to 0$ if the search cost and the firm's discount factor are 0. For the slope of the optimal price to be bounded from zero, the firm must discount the future if there is no search cost. In this section, we consider the case with a positive search cost. In the presence of a positive search cost, the continuation value of searching may be negative, hence both the purchase and exit boundaries are finite, giving two unknowns to be determined at any given time in the optimal search problem.

Since the optimal price is non-stationary with K bounded from zero in the presence of search friction, even if the firm is perfectly patient, we will focus on the no-discounting case in this section.[3] We consider K in the vicinity of 0, and examine whether the firm would benefit from slightly increasing ($K \gtrsim 0$) or decreasing ($K \lesssim 0$) the price over time. The consumer optimal response to the linear pricing $t \mapsto p_0 + \sqrt{\varepsilon}Kt$ is characterized by the moving purchase and exit boundaries $\bar{V}_t$, and $\underline{V}_t$ as in Proposition 3, in particular we have

$$\bar{V}_0 = p_0 + \bar{V}[\sqrt{\varepsilon}K], \qquad \underline{V}_0 = p_0 + \underline{V}[\sqrt{\varepsilon}K],$$

where $\bar{V}[\sqrt{\varepsilon}K], \underline{V}[\sqrt{\varepsilon}K]$ depends on $\sqrt{\varepsilon}K$ and are determined in Proposition 3.

Proposition 5. *Suppose the search cost is positive $c > 0$, and the firm is perfectly patient $m = 0$. The firm's expected profit from a consumer whose initial valuation is x is the following.*

$$\begin{aligned}\mathcal{V}^S(x; p_0, K) = &\frac{p_0 - g + (\bar{V}_0 + x - 2\underline{V}_0)}{1 - \exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)} - \frac{2(\bar{V}_0 - \underline{V}_0)}{\left(1 - \exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)\right)^2} \\ &- \frac{(p_0 - g + (\bar{V}_0 - x))\exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^2}(x - \underline{V}_0)\right)}{1 - \exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)} + \frac{2(\bar{V}_0 - \underline{V}_0)\exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^2}(x - \underline{V}_0)\right)}{\left(1 - \exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)\right)^2}\end{aligned} \tag{13}$$

[3] As we can see from the zero search cost case, the firm is more inclined to charge non-stationary prices if it discounts the future.

if $x \in (\bar{V}_0, \underline{V}_0)$ *and* $K \neq 0$, *and* $\mathcal{V}^S(x;p_0,K) = (p_0 - g)\left(\frac{x-\underline{V}_0}{\bar{V}_0-\underline{V}_0}\right)$ *if* $x \in (\bar{V}_0, \underline{V}_0)$ *and* $K = 0$. $\mathcal{V}^S(x;p_0,K) = 0$ *if* $x \leq \underline{V}_0$. $\mathcal{V}^S(x;p_0,K) = p_0 - g$ *if* $x \geq \bar{V}_0$.

The firm's value function in the above proposition allows us to characterize under what conditions the firm intends to increase the price over time and under what conditions the firm seeks to decrease the price over time. By keeping $K \sim 0$ it is also automatically optimal to set p_0 to be the optimal static price according to Branco et al. (2012):

$$\hat{p} = \hat{p}(x) = \begin{cases} \frac{x+g-\underline{V}}{2}, & \underline{V} + g < x < 2\bar{V} - \underline{V} + g \\ x - \bar{V}, & x \geq 2\bar{V} - \underline{V} + g \end{cases}.$$

Define $q = \frac{x-\underline{V}-g}{2(\bar{V}-\underline{V})}$ as the initial relative position of the consumer between the purchasing and quitting boundaries under $\hat{p}$. It turns out that

$$\frac{\partial \mathcal{V}^S}{\partial K}(x;p_0 = \hat{p}, K = 0) = \frac{(\bar{V} - \underline{V})^2}{3\sigma^2}(1-2q)q(1-q) - (\bar{S}q + \underline{S}(1-q))q$$

is not identically zero for all $q \in [0,1]$ and its sign depends only on σ^2/r, c/r and q. This shows that for a generic $q \in [0,1]$ the optimal strategy (p_0^*, K^*) is such that K^* must be bounded away from 0 even for $m = 0$. The seller can immediately improve its expected profit by setting $K \gtrsim 0$ if $\frac{\partial \mathcal{V}^S}{\partial K}(q;p_0 = \hat{p}, K = 0) > 0$, and by setting $K \lesssim 0$ if $\frac{\partial \mathcal{V}^S}{\partial K}(q;p_0 = \hat{p}, K = 0) < 0$. We summarize the result in Figure 2.

We divide the figure into four regions.

I (Low incentive to search) When the information is too noisy (low σ^2), the search cost is too high (high c), or the consumer values little about the future (high r), the consumer has a low incentive to search for information. In such cases, the firm needs to give the consumer a high surplus to encourage her to search, which hurts its profit. So, it becomes more attractive for the firm to convince the consumer to purchase the product at the beginning, based on the initial valuation and the expected price trajectory. For any given

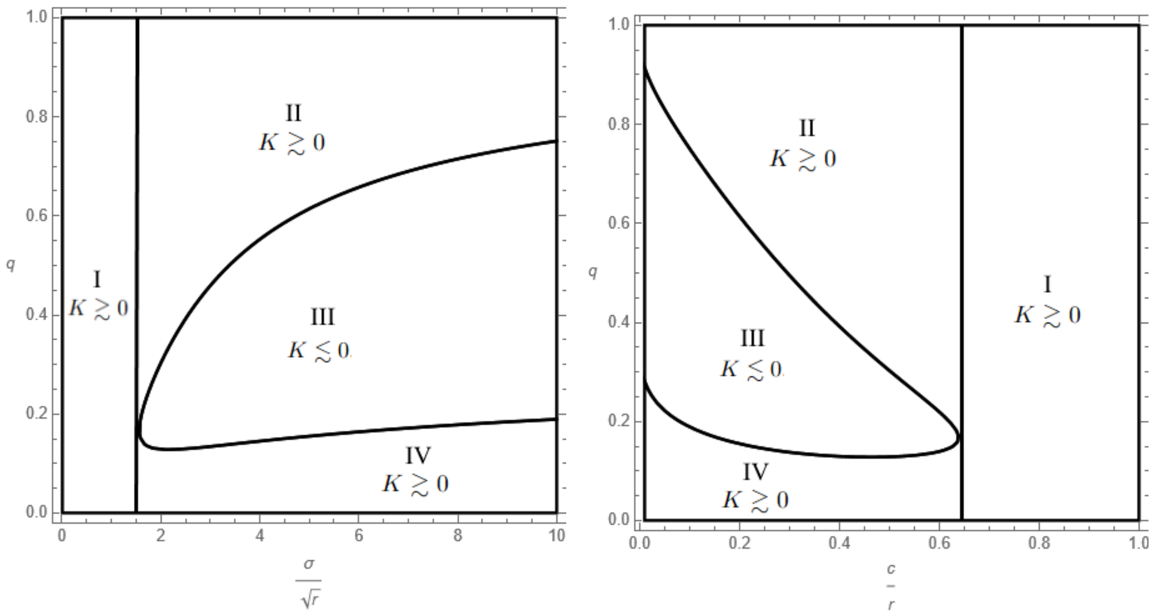

Figure 2: Region plots of seller's expected profit improvement direction of K in the vicinity of $K \sim 0$, using $m = 0$, with $c/r = 1$ for the left plot, and $\sigma^2/r = 1$ for the right plot.

initial price, by charging an increasing price over time, the firm lowers the purchasing threshold at the beginning by making it more desirable for the consumer to make an immediate decision. Compared with the stationary pricing strategy of charging a lower constant price, this non-stationary pricing strategy moves the purchasing threshold in the same direction (downwards) without sacrificing the profit conditional on purchase.

II (High–value consumer) When the consumer has a high initial valuation, she is too valuable to lose from the firm's perspective. Therefore, the firm wants to increase the purchasing probability in this case. Moreover, a high–value consumer can earn a positive payoff from purchasing immediately, which decreases over time because of discounting. Thus, the firm also wants the consumer to buy quickly. An increasing pricing strategy reduces the benefits of searching and encourages the consumer to purchase quickly and with a higher likelihood.

III (Medium–value consumer) When the consumer has a moderate interest in the product, an increase in price does not suffice to convince the consumer to purchase quickly without learning much additional information. Instead, it reduces the benefit of searching because the consumer knows she has to pay a higher price if she learns positive things. Therefore, an increasing price will lead to a quick exit rather than a quick purchase.

The firm can benefit from reducing the price gradually in this case. A decreasing price helps the firm keep the consumer engaged in the search process even if she receives some negative information early on. It increases the purchasing likelihood. Because of the moderate initial valuation, the firm can still obtain a decent profit at a lower price. This pricing strategy protects the firm from missing potentially valuable consumers.

IV (Low–value consumer) By charging a decreasing price over time, the firm can keep the consumer engaged in the search process even if she receives some negative information early on. However, it is not worth it for the firm to reduce the price over time for two reasons. First, the profit from an immediate purchase is already low when the consumer has a low initial valuation. The firm will obtain an even lower profit from purchasing if the consumer searches for a while and eventually buys at a lower price. Second, the consumer must accumulate a lot of positive information before purchasing due to the low initial valuation. The purchasing probability will still be low even if the price slightly reduces over time, and cannot offset the cost of a lower profit per purchase.

The firm quickly filters out many consumers by implementing an increasing pricing strategy instead. On the one hand, the loss from not converting these people is limited due to the low profit per purchase and the low purchasing probability. On the other hand, the benefits of charging a higher price to the remaining consumers are high. Any consumers not quitting despite the increasing price must have learned positive information and are more valuable to the firm.

So far, we discussed K in the vicinity of 0 by analyzing the derivative of $\mathcal{V}^S(x; p_0, K)$ at $K = 0$ with $p_0 = \hat{p} = \frac{x+g-V}{2}$. For any $K \neq 0$, by solving $\frac{\partial \mathcal{V}^S}{\partial p_0}(x; p_0^*, K) = 0$ we find that the optimal initial price p_0 that maximizes $\mathcal{V}^S(x;.,K)$ is:

$$p_0^*(x,K) := \frac{x+g}{2} + \frac{\sigma^2}{2\sqrt{\varepsilon}K} - \frac{\underline{V}[\sqrt{\varepsilon}K]}{2}\left(1 - \coth\frac{\sqrt{\varepsilon}K}{\sigma^2}\left(\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K]\right)\right) \\ - \frac{\bar{V}[\sqrt{\varepsilon}K]}{2}\coth\frac{\sqrt{\varepsilon}K}{\sigma^2}\left(\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K]\right)$$

for $x \in (\underline{V}_0, \bar{V}_0)$ and we can check that $\lim_{K\to 0} p_0^*(x,K) = \hat{p}(x)$.

Lemma 4. *Suppose that $\sigma\sqrt{r} > 0$, then there exists $\lambda_1 > 0$ sufficiently small such that if $(p_0^*, K^*) \in \mathbb{R}\times[-\lambda_1, +\lambda_1]$ is any* local *maximum point of $\mathcal{V}^S(x;.,.)$, then either $p_0^* < \hat{p}, K^* \gtrsim 0$, or $p_0^* > \hat{p}, K^* \lesssim 0$.*

In fact, it is possible to find values of q, r, c, σ^2, and $\lambda_1 > 0$ such that (p_0^*, K^*) is the unique (hence global) maximum point of $\mathcal{V}^S(x;.,.)$ over $\mathbb{R}\times[-\lambda_1, +\lambda_1]$, satisfying either $p_0^* < \hat{p}, K^* \gtrsim 0$, or $p_0^* > \hat{p}, K^* \lesssim 0$.

We note some implications of Lemma 4 beyond the linear pricing strategies. Suppose that we expand the seller's set of admissible pricing strategies to:

$$\mathcal{P}_{\lambda_1,\lambda_2} := \left\{p \in W^{2,\infty}(\mathbb{R}_{>0}) \mid \sup_{t\in\mathbb{R}_{>0}} |p_t'| \leq \lambda_1, \operatorname*{ess\,sup}_{t\in\mathbb{R}_{>0}} |p_t''| \leq \lambda_2\right\}$$

for some $\lambda_1, \lambda_2 > 0$, and by setting $p_0 := \lim_{t\to 0+} p_t$ we interpret $\mathcal{P}_{\lambda_1,\lambda_2} \subset C^1[0,\infty)$. If $\lambda_2 = 0$. Then $\mathcal{P}_{\lambda_1,\lambda_2}$ reduces to the set of linear pricing strategies, however, when $\lambda_2 > 0$ we also allow strategies that are non–linear, but not *too* non–linear. If $\lambda_1 = 0$ then $\mathcal{P}_{\lambda_1,\lambda_2}$ contains only static prices. In practice, sellers may be restricted by some regulations in how fast they can change the price over time, which means $\lambda_1 > 0$ cannot be too large. On the other hand, for any $\delta > 0$ we have:

$$V^S(x) = \sup_{p\in\mathcal{P}_{\lambda_1,\lambda_2}} \left[\mathbb{E}\left[e^{-m\tau}(p_\tau - g)\cdot 1_{v_\tau \geq p_\tau}\cdot 1_{\tau<\delta} | v_0 = x\right] + e^{-m\delta}\int_{\mathbb{R}} \mathcal{V}^S(x; p_{\cdot+\delta}) U(\delta, x) dx\right]. \tag{14}$$

By Assumption 1, we already have that the buyer is myopic and only cares about the p_t and

p'_t at any given time t. Given a finite $\lambda_2 > 0$, we can find $\delta > 0$ such that the first term of (14) can be approximated with a linear pricing $l_t := p_0 + p'_0 \cdot t$ up to an order of some $\varepsilon > 0$, i.e. $\mathbb{E}[e^{-m\tau}(p_\tau - l_\tau) \cdot 1_{v_\tau \geq p_\tau} \cdot 1_{\tau<\delta} | v_0 = x] < \varepsilon/2$. Since survival probability satisfies $\mathbb{P}[\tau^*[l] > \delta] = O\left(\frac{1}{\sigma\sqrt{\delta}}\right)$, we have that the second term of (14) is of $\left(\frac{1}{\sigma\sqrt{\delta}}\right)$–order. Suppose that $\sigma >> 0$, which is relevant in a situation where a large amount of information can be transferred to the buyer effectively, then we may argue that the second term of (14) is $< \varepsilon/2$. In conclusion, for some sufficiently small $\lambda_1 > 0$ and sufficiently large $\sigma^2 >> 0$, Lemma 4 in fact classifies the initial value and initial slope of the optimal pricing strategy over $\mathcal{P}_{\lambda_1,\lambda_2}$.

4.3 Forced–Purchase Strategy

In the previous sections, we focused on the perturbative regime of pricing strategies: linear and slow–moving prices. However, we will show that, in a certain case, an alternative pricing strategy is tractable and leads to interesting results. Namely, when the buyer's initial valuation is sufficiently high, it is optimal for the seller to force an immediate purchase by increasing the price as sharply as possible. This presents another way for the firm to utilize non–stationary pricing to its advantage. In this section, we impose neither the myopic assumption nor the slow-varying price assumption from Assumption 1. The main result of this section is as follows.

Proposition 6. *Let $h \in \mathcal{P}_T$ be an arbitrary pricing strategy strictly monotonically increasing over $[0, T]$ with $h_0 = 0$ and let $p_0 \in \mathbb{R}$ be a constant. Then*

$$\lim_{K\to\infty} \mathcal{V}^S(x; p_0 + Kh) = \begin{cases} p_0 - g, & \text{if } x > p_0 \\ 0, & \text{if } x \leq p_0 \end{cases}. \tag{15}$$

Further, let $\tau^*[p] \in \mathcal{T}$ be the ε–optimal buyer's stopping time to the pricing strategy $p \in \mathcal{P}_T$. Then for the given parameters m, r, c, σ^2 such that $\underline{V} > -\infty$ (i.e. $c > 0$), there exists

$\bar{x} := \bar{x}(m, r, c, \sigma^2)$ such that if $x > g + \bar{x}$ then

$$V^S(x) = \sup_{p \in \mathcal{P}_T} \mathcal{V}^S(x; \tau^*[p], p) = x - g$$

can be approached by the sequence

$$\{p_n := p_{0,n} + K_n h \in \mathcal{P}_T\}_{n \in \mathbb{Z}_{\geq 0}} \tag{16}$$

where $p_{0,n} \to x-$ and $K_n \to +\infty$.

We note that for $x > g + \bar{x}(m, r, c, \sigma^2)$, $c > 0$, we have $V^S(x) = x - g$ regardless of how large $T > 0$ is chosen. The condition $c > 0$ is important as we recall the $c = 0, m = 0$ case with $T \to \infty$ from §4.2 that the optimal initial price is unbounded, leading to $V^S(x) = \infty$. Even if we require $p_0 = x-$, the optimal linear pricing strategy is to set the slope $K \gtrsim 0$ as close to 0 as possible and achieve $\mathcal{V}^S(x; p_0 = x-, K) \sim x - g + \frac{\sigma}{\sqrt{2r}} > x - g$. Although the second inequality in (38) continues to hold even if $\underline{V}_t[p] = -\infty < g$, since the buyer never exits without the exit boundary, the real reason the seller can achieve an expected payoff higher than $x - g$ is because Martingale stopping theorem fails as $v_{t \wedge \tau^*[p]}$ is not bounded in this case.

It's also interesting to compare this section's result with the result from the constant pricing strategy when $c > 0$. We recall the optimal constant price $\hat{p}$ is $\hat{p}(x) = x - \bar{V}$ for all $x \geq 2\bar{V} - \underline{V} + g$, giving the seller's payoff of $x - g - \bar{V} < x - g$. Intuitively, to implement a constant pricing version of forced–purchase strategy, the seller can only set the price as high as $p_0 + \bar{V}$ is still $\leq x$, whereas the purchase boundary itself can be moved with non–stationary pricing strategy which allows for the seller to set a higher price. Therefore, we have demonstrated that by allowing for non–stationary pricing, the seller is able to increase its profit by $\bar{V}$. The results for linear pricing in §4.2, in particular Figure 2, also suggests the seller's expected profit will increase with increasing price strategy when q is high. Indeed, both analytic plots of $\mathcal{V}^S(x; p_0, K)$ and some experimentation with our backward–induction–based simulations both appeared to agree that $K \gtrsim 0$ in region II of Figure 2 suggests the implementation of the forced–purchase strategy. Although it is tempting to speculate that the boundary between region II and III in Figure 2 is exactly $\bar{x}$, we do not believe this is true, however their relationship certainly posts an interesting direction for future investigation.

5 Conclusion

Consumers gather information gradually to help themselves make the purchasing decision. The increasingly popular privacy regulations have made it harder for firms to track individuals in real time. Even if a firm can track consumers' browsing behavior, it is difficult for the firm to infer whether consumers like the information they see. Without the ability to track consumer's valuation evolution about the product, one may think that firms can only offer a constant price. The major innovation of our paper is to allow the price to be a function of time rather than the consumer's current valuation of the product. We find that constant price is not always optimal for the firm. It can benefit from using non-stationary pricing strategies. By assuming that the consumer is sufficiently myopic and uses $\varepsilon-$ optimal strategies, and that the firm uses linear prices that vary slowly, we show that, when the search cost is zero, the optimal price is arbitrarily close to a constant price if the firm is perfectly patient. In contrast, the slope of the optimal price is bounded from zero if the firm discounts the future. When the search cost is positive, the optimal price is non-stationary, even if the firm is perfectly patient. In particular, the firm always increases the price over time if the information is too noisy or the search cost is too high. In other cases where consumers have a stronger incentive to search, the firm charges an increasing price for consumers with high or low initial valuation, while charging a decreasing price for medium–value consumers.

This paper makes two main contributions. On the one hand, it provides new managerial insights for the firm by considering non-stationary pricing. The primary goal of marketing is to reduce the cost and increase the return. Using time as the information source to guide pricing decisions is essentially free. Firms do not need to invest heavily in the tracking technology. Hence, all the increased revenue due to non-stationary pricing becomes profit. It is also immune to privacy regulations. Regulations can prevent firms from tracking consumers' demographic information, browsing behavior, and other characteristics, but cannot ban the time to which everyone has access. Extant research mainly focuses on the economic impact of privacy regulations. We contribute to this stream of literature by studying what firms can do to respond to such privacy regulations.

On the other hand, our non-stationary pricing framework and solution method contribute theoretically to optimal control. The vast majority of papers in marketing and economics restrict attention to Markov strategies. The most common reason is tractability rather than managerial justifications. Therefore, this restriction may not be without loss of generality and may cost firms "free dollars," as shown in this paper. We view this paper as the first step in understanding firms' non-Markov interventions in the presence of consumer search.

Appendix

Proof of Lemma 1.

Part 1: The fact that $V^B(T,x) = V_0^B(x;p_T)$ is clear. For x sufficiently high (e.g. higher than the purchase boundary of $V_0^B(.;\max_{t\in[0,T]} p_t)$) we must have that $V^B(t,x) = x - p_t$. Likewise, for x sufficiently low (e.g. lower than the exit boundary of $V_0^B(.;\min_{t\in[0,T]} p_t)$). In particular, $V^B(t,.)$ must also satisfies the value–matching conditions at some boundaries $\bar{V}_t, \underline{V}_t : [0,T] \to \mathbb{R}, \bar{V}_t > \underline{V}_t$.

For any fixed t we argue $V^B(t,.)$ is monotonically increasing and convex. At x and for an arbitrary small $\varepsilon > 0$ we can find $\tau_{t,x,\varepsilon} \in \mathcal{T}$ such that $\mathcal{V}^B(t,x;\tau_{t,x,\varepsilon},p) \geq V^B(t,x) - \varepsilon$. Fixing $\tau_{t,x,\varepsilon}$, then we note that $\mathcal{V}^B(t,x+\Delta;\tau_{t,x,\varepsilon},p) \geq V^B(t,x) - \varepsilon + m_{t,x,\varepsilon}\Delta$ where $m_{t,x,\varepsilon} := \mathbb{E}\left[e^{-r(\tau_{t,x,\varepsilon}-t)}|v_t = x\right] \geq 0$. By the non–optimality of $\tau_{t,x,\varepsilon}$ at $x+\Delta$ we have that $V^B(t,x+\Delta) \geq V^B(t,x) - \varepsilon + m_{x,\varepsilon}\Delta$. Since ε is arbitrary small, we conclude that $V^B(t,x)$ is convex in x, and since $m_{t,x,\varepsilon} \geq 0$ we have that $V^B(t,x)$ is monotonically increasing in x. It then follows that $V^B(t,.)$ is continuous and differentiable a.e. with $|\partial_x V^B| \leq 1$ (hence Lipschitz continuous in x) and with weak derivative $\partial_x V^B(t,.) \in L^\infty_{loc}(\mathbb{R})$ (see §5.8.2 of Evans (2022)).

For any fixed x we show that $V^B(.,x)$ is differentiable a.e.. Given $t,t' \in [0,T]$, by appropriately relabeling $t > t'$ or $t < t'$, we can assume that $V^B(t',x) \leq V^B(t,x)$. Let $\tau_{t,x,\varepsilon} \in \mathcal{T}$ be defined as above, the expected payoff from keep using $\tau_{t,x,\varepsilon}$ at t' (which is the same as shifting p back by $\Delta := t' - t$, keeping t and the stopping time fixed):

$$\mathcal{V}^B(\Delta) := \mathcal{V}^B(t,x;\tau_{t,x,\varepsilon},p_{.-\Delta})$$

is continuously differentiable in Δ. Clearly, $|(\mathcal{V}^B)'(\Delta)| \leq K := \max_{s\in[0,T]} |p'_s|$ and so $V^B(t',x) \geq V^B(t,x) - \varepsilon - K|\delta t|$. Since $\varepsilon > 0$ is arbitrary small, we have that $|V^B(t',x) - V^B(t,x)| \leq K|t'-t|$. This means $V^B(.,x)$ is Lipschitz continuous and hence differentiable a.e. with bounded weak derivative $\partial_t V^B(.,.) \in L^\infty([0,T]\times\mathbb{R})$, $|\partial_t V^B| \leq K$

(see §5.8.2 of Evans (2022)).

It follows from the argument above that for any fixed t, the set $U_t := \{x \in \mathbb{R} | V^B(t,x) > \max\{x - p_t, 0\}\}$ is an open interval which can be written in the form $(\underline{V}_t, \bar{V}_t)$ for some continuous functions $\bar{V}_t, \underline{V}_t : [0,T] \to \mathbb{R}, \bar{V}_t > \underline{V}_t$.

We show that $\partial_x V^B(t,.)$ is continuous. We already knew that $\partial_x V^B(t,.)$ is monotonically increasing, therefore for any $x^* \in \mathbb{R}$ we have

$$\partial_x V^B(t, x^{*+}) := \lim_{x' \to x^{*+}} \partial_x V^B(t, x') \geq \lim_{x' \to x^{*-}} \partial_x V^B(t, x') =: \partial_x V^B(t, x^{*-}),$$

and at $(t - \delta t, x^*)$ the consumer is forced to continues learning for δt, before continue optimally under τ^*, then the expected payoff is:

$$\begin{aligned}
e^{-r\delta t} & \mathbb{E}\left[V^B(t, v_t) | v_t \geq x^*\right] \mathbb{P}\left[v_t \geq x^* | v_{t-\delta t} = x^*\right] \\
& + e^{-r\delta t} \mathbb{E}\left[V^B(t, v_t) | v_t < x^*\right] \mathbb{P}\left[v_t < x^* | v_{t-\delta t} = x^*\right] - \frac{c}{r}\left(1 - e^{-r\delta t}\right) \\
& \geq \frac{1}{2}\left[V^B(t - \delta t, x^*) + \partial_x V^B(t, x^{*+}) \cdot \frac{\sigma\sqrt{2\delta t}}{\sqrt{\pi}}\right] \\
& + \frac{1}{2}\left[V^B(t - \delta t, x^*) - \partial_x V^B(t, x^{*-}) \cdot \frac{\sigma\sqrt{2\delta t}}{\sqrt{\pi}}\right] + O(\delta t) \\
& = V^B(t - \delta t, x^*) + \left(\partial_x V^B(t, x^{*+}) - \partial_x V^B(t, x^{*-})\right) \cdot \frac{\sigma\sqrt{\delta t}}{\sqrt{2\pi}} + O(\delta t).
\end{aligned}$$

The inequality followed from the convexity and $V^B(t - \delta t, x^*) = V^B(t, x^*) + O(\delta t)$ by the continuity in t. We can see that the payoff is $> V^B(t - \delta t, x^*)$ for sufficiently small δt unless $\partial_x V^B(t, x^{*+}) = \partial_x V^B(t, x^{*-})$. Therefore, $V^B(t,.)$ is in fact continuously differentiable for each fixed t. The smooth–pasting conditions also follow.

Now, consider any $(t,x) \in \Omega$. From the principle of optimality, we have for any $t' \geq t$:

$$V^B(t,x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-r(\tau \wedge t' - t)} V^B(\tau \wedge t', v_{\tau \wedge t'}) - \int_t^{\tau \wedge t'} c e^{-r(s-t)} ds | v_t = x\right].$$

For any $\varepsilon' > 0$ and $\delta t > 0$ it is possible to find $\tau_{t,x,\varepsilon'} \in \mathcal{T}$ and because V^B is Lipschitz continuous it is also possible to find $\varepsilon > 0$ independent of (t, x) such that

$$\mathbb{E}\left[e^{-r\tau_{t,x,\varepsilon'}\wedge\delta t}V^B(t' + \tau_{t,x,\varepsilon'} \wedge \delta t, x' + \sigma W_{\tau_{t,x,\varepsilon'}\wedge\delta t}) - \frac{c}{r}\left(1 - e^{-r\tau_{t,x,\varepsilon'}\wedge\delta t}\right)\right] + \varepsilon'\delta t$$
$$\geq V^B(t', x') \geq \mathbb{E}\left[e^{-r\delta t}V^B(t' + \delta t, x' + \sigma W_{\delta t}) - \frac{c}{r}\left(1 - e^{-r\delta t}\right)\right]$$

for all $(t', x') \in (t, x) + [-\varepsilon, +\varepsilon]^2$. The second inequality followed simply from the non–optimality. The usual way to proceed is by applying Ito's Lemma. Although V^B is not known to be $C^{1,2}(\Omega)$, more general versions of Ito's Lemma are available such as the one for convex functions (see §3.6 Karatzas and Shreve (2012)), or the weak derivative version Aebi (1992). In any case, the standard treatment is by mollifying the problematic function which we shall provide detail for our simple case for completeness.

For any $\varepsilon > 0$ the mollification of any function $f \in L^1_{loc}([0, T] \times \mathbb{R})$ is the smooth function $f_\varepsilon := \eta_\varepsilon * f$, where $\eta_\varepsilon(t, x) \propto e^{-\frac{1}{1-\|(t,x)\|_2/\varepsilon^2}} 1_{(t,x)\in[-\varepsilon,+\varepsilon]^2}$ is a standard compactly supported bump–function. Applying the mollification in (t', x') centered at (t, x) to the above inequality before applying Ito's Lemma we obtain:

$$\mathbb{E}\Bigg[\int_t^{t+\tau_{t,x,\varepsilon'}\wedge\delta t} e^{-r(s-t)}$$
$$\times\left(\frac{\sigma^2}{2}\partial_x^2 V_\varepsilon^B(s, v_s) + \partial_t V_\varepsilon^B(s, v_s) - rV_\varepsilon^B(s, v_s) - c\right) ds|v_t = x\Bigg] + \varepsilon'\delta t$$
$$\geq 0 \geq \mathbb{E}\left[\int_t^{t+\delta t} e^{-r(s-t)}\left(\frac{\sigma^2}{2}\partial_x^2 V_\varepsilon^B(s, v_s) + \partial_t V_\varepsilon^B(s, v_s) - rV_\varepsilon^B(s, v_s) - c\right) ds|v_t = x\right].$$

Taking the limit $\delta t \to 0$ we get

$$\left|\frac{\sigma^2}{2}\partial_x^2 V_\varepsilon^B(t, x) + \partial_t V_\varepsilon^B(t, x) - rV_\varepsilon^B(t, x) - c\right| \leq \varepsilon'.$$

Recall that $\varepsilon' > 0$ arbitrary and $\varepsilon > 0$ is independent of (t, x), this establishes a uniform convergence $\frac{\sigma^2}{2}\partial_x^2 V_\varepsilon^B + \partial_t V_\varepsilon^B - rV_\varepsilon^B - c \to 0$ on Ω as $\varepsilon \to 0$. Let $g := \partial_t V^B - rV^B - c$. Since $|V^B| \le L := \max_{s\in[0,T]}(\bar{V}_s - p_s)$ and $|\partial_t V^B| \le K := \max_{s\in[0,T]} |p'_s|$, therefore g is bounded: $|g(t,x)| \le K + rL + c$. It follows that $g(t,.) \in L^\infty_{loc}(\mathbb{R}) \subset L^1_{loc}(\mathbb{R})$. From the properties of mollification (see §5.3.1 of Evans (2022)) we already know the $L^1_{loc}(\mathbb{R})$ convergence $\partial_t V_\varepsilon^B(t,.) - rV_\varepsilon^B(t,.) - c \to g(t,.)$, it follows from the uniform convergence above that $\partial_x^2 V_\varepsilon^B(t,.) \to g(t,.)$ in $L^1_{loc}(\mathbb{R})$ as $\varepsilon \to 0$. Therefore the weak derivative $\partial_x^2 V^B(t,.)$ exists, coincides with $g(t,.)$ a.e.,

$$\frac{\sigma^2}{2}\partial_x^2 V^B + \partial_t V^B - rV^B - c = 0 \quad \text{a.e. on } \Omega$$

and $\partial_x^2 V^B(t,.) \in L^\infty(\mathbb{R})$ for each t.

Part 2: Let V be the weak solution as given in Lemma's statement but for convenience, let's also extend the definition by $V(t,x) := \max\{x - p_t, 0\}$ for $(t,x) \notin \Omega$, and $V(t \ge T, x) := V_0^B(x; p_T)$. As before, we consider the mollification $V_\varepsilon := \eta_\varepsilon * V$ before proceeding with Ito's Lemma. For any given stopping time $\tau \in \mathcal{T}$ we have that

$$\begin{aligned}\mathbb{E}\left[e^{-r\tau} V_\varepsilon(\tau, v_\tau) | v_t = x\right] - e^{-rt} V_\varepsilon(t,x) \\ = \mathbb{E}\left[\int_t^\tau e^{-rs}\left(-rV_\varepsilon(s,v_s) + \partial_t V_\varepsilon(s,v_s) + \frac{\sigma^2}{2}\partial_x^2 V_\varepsilon(s,v_s)\right) ds | v_t = x\right].\end{aligned} \tag{17}$$

Since we have assumed $p_t = p_T$ constant for $t \ge T$, we may restrict our attention only to $\tau \in \mathcal{T}$ which coincides with the constant price stopping rule (characterized by the purchase and exit boundaries $p_T + \bar{V}, p_T + \underline{V}$) whenever $\tau \ge T$. In other words, we only consider any arbitrarily stopping time τ which is only potentially non–optimal for $0 \le \tau < T$.

We proceed by taking the limit as $\varepsilon \to 0$ on both–sides of (17). In the following, let's assume that $\varepsilon < \bar{\varepsilon}$ for some $\bar{\varepsilon} > 0$. For the LHS, we have a pointwise convergence: $V_\varepsilon \to V$ over $[0,T] \times \mathbb{R}$. Let $L := \max_{s\in[0,T]}(\bar{V}_s - p_s)$ then $|V_\varepsilon(t,x)| \le \max\{x - p_t, L\}$. Further,

$$\mathbb{E}\left[e^{-r\tau}\max\{v_\tau - p_\tau, L\}|v_t = x\right] \le V_0^B(x; p = \min_{t\ge 0} p_t, c = 0) + L$$
$$\le \max\left\{x - \min_{t\ge 0} p_t, \frac{\sigma}{\sqrt{2r}}\right\} + L < \infty$$

and so $\mathbb{E}\left[e^{-r\tau}V_\varepsilon(\tau, v_\tau)|v_t = x\right] \to \mathbb{E}\left[e^{-r\tau}V(\tau, v_\tau)|v_t = x\right]$ by the Dominated Convergence Theorem. We now turn to the RHS. Since V is a weak solution, we have $\frac{\sigma^2}{2}\partial_x^2 V_\varepsilon + \partial_t V_\varepsilon - rV_\varepsilon - c = 0$ for all $(t, x) \in \{\underline{V}_t + \varepsilon < x < \bar{V}_t - \varepsilon, t \in [0, T]\}$, so let's focus on the vicinity of $x = \bar{V}_t$ boundary, the $x = \underline{V}_t$ boundary is similar. By the convexity of $V(t, .)$ we have that the weak derivative $\partial_x^2 V(t, .)$ coincides a.e. with the classical second derivative, hence $\partial_x^2 V(t, .) \ge 0$ a.e. and so $\partial_x^2 V_\varepsilon(t, x) \ge 0$. Since $V \in L^\infty(0, T; W^{2,\infty}_{loc}(\mathbb{R}))$ implies the uniform essential boundedness of $\partial_x^2 V(t, .)$ for $t \in [0, T]$, hence for any given $\bar{\delta} > 0$, let $\bar{\varepsilon} > 0$ be such that $1 \ge \partial_x V_\varepsilon(t, x) > 1 - \bar{\delta}$ for all $(t, x) \in \{\bar{V}_t - \bar{\varepsilon} < x < \bar{V}_t + \bar{\varepsilon}, t \in [0, T]\}$ and all $\varepsilon < \bar{\varepsilon}$. For $\delta > 0$ let's define $\bar{V}_{t,\delta,\varepsilon} := \min\{x|V_\varepsilon(t, x) = x - p_t + \delta\}$ which is differentiable in t. For any $t \in [0, T]$ and $\bar{V}_t > x > \bar{V}_t - \bar{\varepsilon}$ we can choose $\delta > 0$ such that $\bar{V}_{t,\delta,\varepsilon} = x$ and it follows from differentiating the defining equation $V_\varepsilon(t, \bar{V}_{t,\delta,\varepsilon}) = \bar{V}_{t,\delta,\varepsilon} - p_t + \delta$ that

$$\partial_t V_\varepsilon(t, \bar{V}_{t,\delta,\varepsilon}) + \bar{V}'_{t,\delta,\varepsilon}\partial_x V_\varepsilon(t, \bar{V}_{t,\delta,\varepsilon}) = \bar{V}'_{t,\delta,\varepsilon} - p'_t$$
$$\implies |\partial_t V_\varepsilon(t, x) + p'_t| < |\bar{V}'_{t,\delta,\varepsilon}| \cdot |1 - \partial_x V_\varepsilon(t, x)| < M\bar{\delta},$$

where $M := \max_{s\in[0,T]} |p'_s| + \operatorname{ess\,sup}_{[0,T]\times\mathbb{R}} |\partial_t V|$. The argument we have been through above shows that for (t, x) in the $\bar{\varepsilon}$ vicinity of the boundary $\partial\bar{\Omega}$ we have $\partial_t V_\varepsilon$ varies no more than $M\bar{\delta}$, whereas $\partial_x^2 V_\varepsilon$ decreases rapidly to zero away from Ω. Therefore, $\frac{\sigma^2}{2}\partial_x^2 V_\varepsilon + \partial_t V_\varepsilon - rV_\varepsilon - c - M\bar{\delta} < 0$ for $(t, x) \notin \{\underline{V}_t + \varepsilon < x < \bar{V}_t - \varepsilon, t \in [0, T]\}$. It follows that for all $\varepsilon > 0$ with $\varepsilon < \bar{\varepsilon}$ the RHS of (17) is always $\le \mathbb{E}\left[\int_t^\tau ce^{-rs}ds|v_t = x\right] + M\bar{\delta}$. Since $\bar{\delta} > 0$ is arbitrary small, under the limit $\varepsilon \to 0$ (17) becomes

$$\mathbb{E}\left[e^{-r\tau}V(\tau, v_\tau)|v_t = x\right] - e^{-rt}V(t, x) \le \mathbb{E}\left[\int_t^\tau ce^{-rs}ds|v_t = x\right],$$

Using $V(t, v_t) \geq \max\{v_t - p_t, 0\}$ and rearranging the inequality above, we obtain:

$$V(t,x) \geq \mathbb{E}\left[e^{-r(\tau-t)}\max\{v_\tau - p_\tau, 0\} - \int_t^\tau ce^{-r(s-t)}ds | v_t = x\right]. \tag{18}$$

Since τ is arbitrary, we have by the definition of supremum that $V(t,x) \geq V^B(t,x)$.

For any $(t,x) \in \Omega$, consider the hitting time $\tau^* \in \mathcal{T}$ of the valuation process starting at x to the boundary $\partial\bar{\Omega}$. Any sample path of v_t from x to $v_{\tau^*} \in \partial\bar{\Omega}$ is contained entirely inside Ω where $\frac{\sigma^2}{2}\partial_x^2 V + \partial_t V - rV - c = 0$ a.e.. Take the $\varepsilon \to 0$ limit of (17), use the boundedness of each term on the RHS's integrand over $\bar{\Omega}$ to obtain the convergence of RHS's integral via Dominated Convergence Theorem, and the fact $V(\tau^*, v_{\tau^*}) = \max\{v_{\tau^*} - p_{\tau^*}, 0\}$ to conclude that we get (18) with equality. Therefore, the supremum can be reached with τ^*, hence $V(t,x) = V^B(t,x)$. □

Proof of Lemma 2. According to Lemma 1, the solution V must coincide with the value function V^B on $[0,T] \times \mathbb{R}$, therefore, we may take $V(.,.;p)$ and $V(.,.;q)$ to be given by (3) with p and q, respectively. Let's consider a fixed $(t,x) \in [0,T] \times \mathbb{R}$, and let's suppose that $V(t,x;q) \leq V(t,x;p)$. For an arbitrary $\varepsilon > 0$, let $\tau_{t,x,\varepsilon}[p] \in \mathcal{T}$ be such that $\mathcal{V}^B(t,x;\tau_{t,x,\varepsilon}[p],p) \geq V(t,x;p) - \varepsilon$, then

$$\begin{aligned} V(t,x;q) \geq \mathcal{V}^B(t,x;\tau_{t,x,\varepsilon}[p],q) > \mathcal{V}^B(t,x;\tau_{t,x,\varepsilon}[p];p) - \max_{s\in[t,T]} e^{-r(s-t)}|p_s - q_s| \\ \geq V(t,x;p) - \max_{s\in[t,T]} e^{-r(s-t)}|p_s - q_s| - \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it must be the case that:

$$V(t,x;p) \geq V(t,x;q) \geq V(t,x;p) - \max_{s\in[t,T]} e^{-r(s-t)}|p_s - q_s|.$$

If $V(t,x;q) \geq V(t,x;p)$, then we simply switch the role of p, q and follow through with the above argument, hence we get that

$$|V(t,x;p) - V(t,x;q)| \leq \max_{s\in[t,T]} e^{-r(s-t)}|p_s - q_s|,$$

which proves the result. $\square$

Proof of Proposition 1. By Lemma 2 we know that $V^B(.,.;\tilde{p})$ would agree with $V^B(.,.;p)$ up to the $\sqrt{\varepsilon}$ order, therefore we propose the following ansatz:

$$V^B(t,x;\tilde{p}) = V^B(t, x - \sqrt{\varepsilon}Kh_t; p) + \sqrt{\varepsilon}V_1^B(t,x) + O(\varepsilon).$$

We note that $V^B(t, x - \sqrt{\varepsilon}Kh_t; p)$ is simply the solution $V^B(t,x;p)$ shifted according to $\sqrt{\varepsilon}Kh$ which satisfies the value–matching and smooth–pasting conditions at $\bar{V}[p] + \sqrt{\varepsilon}Kh$ and $\underline{V}[p] + \sqrt{\varepsilon}Kh$, but does not satisfies the PDE, hence the $\sqrt{\varepsilon}V_1^B$ correction is needed. At $t = 0$, $V^B(0, x - \sqrt{\varepsilon}Kh_0; p)$ can also be recognized as $V^B(0, x; \tilde{p}^0)$ where $\tilde{p}^0 := p + \sqrt{\varepsilon}Kh_0$ is a pricing strategy shifted from p by a constant $\sqrt{\varepsilon}Kh_0$. If $K > 0$, then $\tilde{p} \geq \tilde{p}^0$ because h is monotonically increasing, and we have that $V^B(0,x;\tilde{p}) \leq V^B(0,x;\tilde{p}^0)$, i.e. $V_1^B(0,x) \leq 0$. Similarly if $K < 0$, then $V_1^B(0,x) \geq 0$. We can repeat the above argument at any given t, hence for any $(t,x) \in [0,\infty) \times \mathbb{R}$ we conclude that:

$$V_1^B(t,x) \leq 0 \text{ if } K > 0, \quad V_1^B(t,x) \geq 0 \text{ if } K < 0. \tag{19}$$

By adding $\sqrt{\varepsilon}V_1^B$ correction, we further need a $\sqrt{\varepsilon}$–order correction to the purchase and exit boundaries $\bar{V}[p] + \sqrt{\varepsilon}Kh$ and $\underline{V}[p] + \sqrt{\varepsilon}Kh$ which take the form (6). The equation for V_1^B can be found by collecting the $\sqrt{\varepsilon}$–order terms in the PDE of $V^B(.,.;\tilde{p})$:

$$\frac{\sigma^2}{2}\partial_x^2 V_1^B(t,x) + \partial_t V_1^B(t,x) - rV_1^B(t,x) - Kh_t'\partial_x V^B(t,x;p) = 0. \tag{20}$$

We remark that if $h_t' = 0$ for all t, i.e. $\tilde{p}$ is simply a shift by a constant from p, then

$V_1^B = 0$ is the unique solution given all the boundary conditions. To study $\bar{R}$ and $\underline{R}$ we will now analyze the boundary conditions of $V^B(.,.;\tilde{p})$ to the first–order in $\sqrt{\varepsilon}$. Note that $V^B(t, x - \sqrt{\varepsilon}Kh_t; p)$ automatically satisfies the value–matching conditions at $\bar{V}[\tilde{p}]$ and $\underline{V}[\tilde{p}]$, as we will confirm below, because $\partial_x V^B(t, \bar{V}_t[p]; p) = 1$ and $\partial_x V^B(t, \underline{V}_t[p]; p) = 0$. We have via Taylor expansion and comparing the $\sqrt{\varepsilon}$–order terms:

$$
\begin{aligned}
&V^B(t, \bar{V}_t[\tilde{p}]; \tilde{p}) = \bar{V}_t[\tilde{p}] - \tilde{p}_t \\
&\qquad \implies V^B(t, \bar{V}_t[p] + \sqrt{\varepsilon}\bar{R}_t; p) + \sqrt{\varepsilon}V_1^B(t, \bar{V}_t[p]) = \bar{V}_t[p] - p_t + \sqrt{\varepsilon}\bar{R}_t \\
&\qquad\qquad V_1^B(t, \bar{V}_t[p]) = -\bar{R}_t \partial_x V^B(t, \bar{V}_t[p]; p) + \bar{R}_t \implies V_1^B(t, \bar{V}_t[p]) = 0.
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
&\partial_x V^B(t, \bar{V}_t[\tilde{p}]; \tilde{p}) = 1 \implies \partial_x V^B(t, \bar{V}_t[p] + \sqrt{\varepsilon}\bar{R}_t; p) + \sqrt{\varepsilon}\partial_x V_1^B(t, \bar{V}_t[p]) = 1 \\
&\qquad \partial_x V_1^B(t, \bar{V}_t[p]) = -\bar{R}_t \partial_x^2 V^B(t, \bar{V}_t[p]; p) \implies \bar{R}_t = -\frac{\partial_x V_1^B(t, \bar{V}_t[p])}{\partial_x^2 V^B(t, \bar{V}_t[p]; p)}.
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
&V^B(t, \underline{V}_t[\tilde{p}]; \tilde{p}) = 0 \implies V^B(t, \underline{V}_t[p] + \sqrt{\varepsilon}\underline{R}_t; p) + \sqrt{\varepsilon}V_1^B(t, \underline{V}_t[p]) = 0 \\
&\qquad\qquad \implies V_1^B(t, \underline{V}_t[p]) = 0.
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
&\partial_x V^B(t, \underline{V}_t[\tilde{p}]; \tilde{p}) = 0 \implies \partial_x V^B(t, \underline{V}_t[p] + \sqrt{\varepsilon}\underline{R}_t; p) + \sqrt{\varepsilon}\partial_x V_1^B(t, \underline{V}_t[p]) = 1 \\
&\qquad \partial_x V_1^B(t, \underline{V}_t[p]) = -\underline{R}_t \partial_x^2 V^B(t, \underline{V}_t[p]; p) \implies \underline{R}_t = -\frac{\partial_x V_1^B(t, \underline{V}_t[p])}{\partial_x^2 V^B(t, \underline{V}_t[p]; p)}.
\end{aligned} \tag{24}
$$

Since $V^B(t,.;p)$ is convex: $\partial_x^2 V^B(t,.;p) \geq 0$, it follows from (22) and (24) that the sign of $\bar{R}_t$ and $\underline{R}_t$ are the opposite as the sign of $\partial_x V_1^B(t, \bar{V}_t[p])$ and $\partial_x V_1^B(t, \underline{V}_t[p])$, respectively. From (19) we see that $\partial_x V_1^B(t, \bar{V}_t[p]) \geq 0$ if $K > 0$, i.e. $\bar{R}_t \leq 0$, and $\partial_x V_1^B(t, \bar{V}_t[p]) \leq 0$ if $K < 0$, i.e. $\bar{R}_t \geq 0$. Similarly, $\partial_x V_1^B(t, \underline{V}_t[p]) \leq 0$ if $K > 0$, i.e. $\underline{R}_t \geq 0$, and $\partial_x V_1^B(t, \underline{V}_t[p]) \geq 0$ if $K < 0$, i.e. $\underline{R}_t \leq 0$. Finally, we define: $\bar{S}_t := \bar{R}_t/K \leq 0$, and $\underline{S}_t := \underline{R}_t/K \geq 0$. $\square$

Proof of Proposition 2. Without the loss of generality, let's only consider $t = 0$ and h such that $h_0 = 0$, we can always redefine t and shift $\tilde{p}$ by constant otherwise.

Suppose that $\bigcap_{K>0}(\underline{V}_0[\tilde{p}], \bar{V}_0[\tilde{p}])$ is open and containing x, hence $V^B(t = 0, x; \tilde{p}) > 0$ for all $K > 0$. We can find $\tau_{x,\varepsilon} \in \mathcal{T}$ such that $\mathcal{V}^B(0, x; \tau_{x,\varepsilon}, \tilde{p}) \geq V^B(0, x; \tilde{p}) - \varepsilon$ and for any $\varepsilon' > 0$ we can find $\delta > 0$ such that $\mathbb{P}[\tau_{x,\varepsilon} < \delta] < \varepsilon'$ for all $K > 0$. Additionally, from Proposition 1 we know that $\bar{V}_0[p] \geq \bar{V}_0[\tilde{p}]$. Then:

$$\begin{aligned} V^B(0, x; \tilde{p}) &\leq \mathcal{V}^B(0, x; \tau_{x,\varepsilon}, \tilde{p}) + \varepsilon \\ &\leq (1 - \varepsilon')e^{-r\delta}\mathbb{E}\left[\max\left\{\bar{V}_{\tau_{x,\varepsilon}}[p] - p_{\tau_{x,\varepsilon}} - Kh_{\tau_{x,\varepsilon}}, 0\right\} | \tau_{x,\varepsilon} \geq \delta\right] + \varepsilon'\bar{V}_0[p] + \varepsilon. \end{aligned}$$

Since ε' and ε are arbitrarily small, while the first term is zero for all sufficiently large $K >> 0$, we have that $V^B(0, x; \tilde{p}) \leq 0$, a contradiction. Therefore, $\bigcap_{K>0}(\underline{V}_0[\tilde{p}], \bar{V}_0[\tilde{p}])$ contains no open sets, proving $\lim_{K\to+\infty}\left(\bar{V}_0[\tilde{p}] - \underline{V}_0[\tilde{p}]\right) = 0$. In other words, for any $x \neq \tilde{p}_0$ there exists $K >> 0$ sufficiently large such that the continue learning option is sub–optimal and the buyer immediately purchase if $x > \tilde{p}_0$ and exit if $x < \tilde{p}_0$, proving $\bar{V}_0[\tilde{p}] \to \tilde{p}_0+, \underline{V}_0[\tilde{p}] \to \tilde{p}_0-$ as $K \to +\infty$.

Suppose that $K < 0$, consider any $x \in \mathbb{R}$, we note that

$$V^B(0, x; \tilde{p}) \geq \mathcal{V}^B(0, x; \delta, \tilde{p}) \geq e^{-r\delta}\mathbb{E}\left[\max\left\{v_\delta - p_\delta - Kh_\delta, 0\right\} | v_0 = x\right] \geq -e^{-r\delta}(Kh_\delta + p_\delta)$$

where the last term is $> 0 = V^B(0, x; p)$ for all sufficiently negative $K << 0$. In the above, δ denotes the simple policy of stopping exactly at time δ regardless of the valuation, and the first inequality followed from the sub–optimality of δ. Therefore, for all sufficiently negative $K << 0$, we have that it is optimal to not exit: i.e. $\underline{V}_0[\tilde{p}] < x$ for any $x \in \mathbb{R}$, so that $\lim_{K\to-\infty}\underline{V}_0[\tilde{p}] = -\infty$. On the other hand, we already know that $\bar{V}_0[\tilde{p}] \geq \bar{V}_0[p]$ from Proposition 1, proving $\lim_{K\to-\infty}\left(\bar{V}_0[\tilde{p}] - \underline{V}_0[\tilde{p}]\right) = +\infty$. $\square$

Proof of Lemma 3. Without the loss of generality, let $t = 0$, fix an $x \in \mathbb{R}$, and shift the coordinate if necessary so that $p_0 = 0$. Moreover, let's suppose there exists the optimal stopping policies $\tau^*[p], \tau^*[l] \in \mathcal{T}$ such that $V^B(0,x;p) = \mathcal{V}^B(0,x;\tau^*[p],p)$ and $V^B(0,x;l) = \mathcal{V}^B(0,x;\tau^*[l],l)$ are the value functions under the p and l pricing policies, respectively. Let $T > 0$ be such that $\max_{t\in[0,\infty)} e^{-rt}\left|p_t - p_t^T\right| < \varepsilon/4$ where $p^T \in \mathcal{P}_T$ is given by p over $[0, T-\varepsilon]$ and constant for all $t \geq T$. By the bound $\sup_{t\in[0,\infty)} |p'_t| < \lambda_1$, we may take $T := \frac{2}{r}\log(4\lambda_1/\varepsilon)$. By Lemma 2, we have $V^B(0,x;p) \leq V^B(0,x;p^T) + \varepsilon/4$, and also it's easy to see that $\mathcal{V}^B(0,x;\tau,p^T) \leq \mathcal{V}^B(0,x;\tau,p) + \varepsilon/4$ for any $\tau \in \mathcal{T}$, so let's assume without the loss of generality that $p \in \mathcal{P}_T$ hereafter and we will show that $V^B(0,x;p) \leq \mathcal{V}^B(0,x;\tau^*[l],p) + \varepsilon/2$. We have

$$\begin{aligned}V^B(0,x;p) = \mathbb{E}&\left[e^{-r\tau^*[p]}\max\{v_{\tau^*[p]} - p_{\tau^*[p]}, 0\} - \frac{c}{r}\left(1 - e^{-r\tau^*[p]}\right)|v_0 = x, \tau^*[p] < \delta\right]\mathbb{P}[\tau^*[p] < \delta]\\ &+ \mathbb{E}\left[e^{-r\tau^*[p]}\max\{v_{\tau^*[p]} - p_{\tau^*[p]}, 0\} - \frac{c}{r}\left(1 - e^{-r\tau^*[p]}\right)|v_0 = x, \tau^*[p] \geq \delta\right]\mathbb{P}[\tau^*[p] \geq \delta],\end{aligned}$$

similarly, for $V^B(0,x;l)$ with l replacing p everywhere in the expression above, and

$$\begin{aligned}\mathcal{V}^B(0,x;\tau^*[l],p) = \mathbb{E}&\left[e^{-r\tau^*[l]}\max\{v_{\tau^*[l]} - p_{\tau^*[l]}, 0\} - \frac{c}{r}\left(1 - e^{-r\tau^*[l]}\right)|v_0 = x, \tau^*[l] < \delta\right]\mathbb{P}[\tau^*[l] < \delta]\\ &+ \mathbb{E}\left[e^{-r\tau^*[l]}\max\{v_{\tau^*[l]} - p_{\tau^*[l]}, 0\} - \frac{c}{r}\left(1 - e^{-r\tau^*[l]}\right)|v_0 = x, \tau^*[l] \geq \delta\right]\mathbb{P}[\tau^*[l] \geq \delta].\end{aligned}$$

Now, we note that for $\tau < \delta$ we have $\max\{v_\tau - p_\tau, 0\} \leq \max\{v_\tau - l_\tau + \delta\varepsilon/2, 0\} \leq \max\{v_\tau - l_\tau, 0\} + \delta\varepsilon/2$, which implies

$$\begin{aligned}&\mathbb{E}\left[e^{-r\tau^*[p]}\max\{v_{\tau^*[p]} - p_{\tau^*[p]}, 0\} - \frac{c}{r}\left(1 - e^{-r\tau^*[p]}\right)|v_0 = x, \tau^*[p] < \delta\right]\\ &\qquad\leq \mathbb{E}\left[e^{-r\tau^*[p]}\max\{v_{\tau^*[p]} - l_{\tau^*[p]}, 0\} - \frac{c}{r}\left(1 - e^{-r\tau^*[p]}\right)|v_0 = x, \tau^*[p] < \delta\right] + \delta\varepsilon/2.\end{aligned}$$

Similarly, using $\max\{v_\tau - l_\tau, 0\} \leq \max\{v_\tau - p_\tau, 0\} + \delta\varepsilon/2$ for $\tau < \delta$, we have:

$$\mathbb{E}\left[e^{-r\tau^*[l]}\max\{v_{\tau^*[l]} - l_{\tau^*[l]}, 0\} - \frac{c}{r}\left(1 - e^{-r\tau^*[l]}\right)|v_0 = x, \tau^*[l] < \delta\right]$$

$$\leq \mathbb{E}\left[e^{-r\tau^*[l]} \max\{v_{\tau^*[l]} - p_{\tau^*[l]}, 0\} - \frac{c}{r}\left(1 - e^{-r\tau^*[l]}\right) | v_0 = x, \tau^*[l] < \delta\right] + \delta\varepsilon/2.$$

On the other hand, since $\max_{t\in[0,T]} |p'_t| \leq \lambda_1$ we have $|p_t| < \lambda_1 T$ and the purchase boundary at any t cannot be higher than if the price was to decrease at maximum rate $-\lambda_1$ after t. From our study of linear pricing in Proposition 3, we have that

$$\bar{V}[-\lambda_1] \leq \bar{V}[-\lambda_1] - \underline{V}[-\lambda_1] \leq \bar{\bar{V}} := \frac{\lambda_1 + \sqrt{\lambda_1^2 + 2r\sigma^2}}{2r} \log\left(1 + \frac{\sqrt{\lambda_1^2 + 2r\sigma^2}}{c}\right)$$

hence $v_{\tau^*[p]} \leq \lambda_1 T + \bar{\bar{V}}$, and consequently:

$$\begin{aligned}
&\mathbb{E}\left[e^{-r\tau^*[p]} \max\{v_{\tau^*[p]} - p_{\tau^*[p]}, 0\} - \frac{c}{r}\left(1 - e^{-r\tau^*[p]}\right) | v_0 = x, \tau^*[p] \geq \delta\right] \\
&\qquad \leq e^{-r\delta}\left(2\lambda_1 T + \bar{\bar{V}} + c/r\right) - \frac{c}{r} < (1-\delta)\varepsilon/4 - \frac{c}{r} \\
&\leq \mathbb{E}\left[e^{-r\tau^*[p]} \max\{v_{\tau^*[p]} - l_{\tau^*[p]}, 0\} - \frac{c}{r}\left(1 - e^{-r\tau^*[p]}\right) | v_0 = x, \tau^*[p] \geq \delta\right] + (1-\delta)\varepsilon/4.
\end{aligned}$$

We can show the similar inequality with l instead of p as follows. If $p'_0 < 0$ then the purchase boundary is decreasing with time, at the rate no faster than $-\lambda_1$, which means $v_{\tau^*[l]} \leq \bar{\bar{V}}$. Also we have that $l_{\tau^*[l]} e^{-r\tau^*[l]} \leq \max_{t\in[0,\infty)} |p'_0| t \cdot e^{-rt} \leq \lambda_1 e^{-1}/r$. Therefore, we have

$$\begin{aligned}
&\mathbb{E}\left[e^{-r\tau^*[l]} \max\{v_{\tau^*[l]} - l_{\tau^*[l]}, 0\} - \frac{c}{r}\left(1 - e^{-r\tau^*[l]}\right) | v_0 = x, \tau^*[l] \geq \delta\right] \\
&\qquad \leq e^{-r\delta}\left(\bar{\bar{V}} + c/r\right) + \frac{\lambda_1 e^{-1}}{r} - \frac{c}{r} < (1-\delta)\varepsilon/4 - \frac{c}{r} \\
&\leq \mathbb{E}\left[e^{-r\tau^*[l]} \max\{v_{\tau^*[l]} - p_{\tau^*[l]}, 0\} - \frac{c}{r}\left(1 - e^{-r\tau^*[l]}\right) | v_0 = x, \tau^*[l] \geq \delta\right] + (1-\delta)\varepsilon/4.
\end{aligned}$$

For the case $p'_0 \geq 0$, we must have that the consumer's expected payoff cannot be better than if the price was to stay constant at $p_0 = 0$, and in the constant price case we have $v_{\tau^*[p_0]} = \bar{V} < \bar{\bar{V}}$. Continue using $l_{\tau^*[l]} e^{-r\tau^*[l]} \leq \lambda_1 e^{-1}/r$, we have:

$$\mathbb{E}\left[e^{-r\tau^*[l]} \max\{v_{\tau^*[l]} - l_{\tau^*[l]}, 0\} - \frac{c}{r}\left(1 - e^{-r\tau^*[l]}\right) | v_0 = x, \tau^*[l] \geq \delta\right]$$

$$\leq e^{-r\delta}(\bar{\bar{V}} + c/r) + \frac{\lambda_1 e^{-1}}{r} - \frac{c}{r} < (1-\delta)\varepsilon/4 - \frac{c}{r}$$
$$\leq \mathbb{E}\left[e^{-r\tau^*[l]} \max\{v_{\tau^*[l]} - p_{\tau^*[l]}, 0\} - \frac{c}{r}\left(1 - e^{-r\tau^*[l]}\right) | v_0 = x, \tau^*[l] \geq \delta\right] + (1-\delta)\varepsilon/4.$$

Putting everything back into the expression for $V^B(0,x)$ we had earlier yields:

$$\begin{aligned}
V^B(0,x;p) &\leq \mathbb{E}\left[e^{-r\tau^*[p]} \max\{v_{\tau^*[p]} - l_{\tau^*[p]}, 0\} - \frac{c}{r}\left(1 - e^{-r\tau^*[p]}\right) | v_0 = x, \tau^*[p] < \delta\right] \mathbb{P}[\tau^*[p] < \delta] \\
&+ \mathbb{E}\left[e^{-r\tau^*[p]} \max\{v_{\tau^*[p]} - l_{\tau^*[p]}, 0\} - \frac{c}{r}\left(1 - e^{-r\tau^*[p]}\right) | v_0 = x, \tau^*[p] \geq \delta\right] \mathbb{P}[\tau^*[p] \geq \delta] \\
&+ \delta(\varepsilon/4)\mathbb{P}[\tau^*[p] < \delta] + (1-\delta)(\varepsilon/4)\mathbb{P}[\tau^*[p] \geq \delta] \\
&\leq \mathcal{V}^B(0,x;\tau^*[p], l) + \varepsilon/4 \leq V^B(0,x;l) + \varepsilon/4 = \mathcal{V}^B(0,x;\tau^*[l], l) + \varepsilon/4 \\
&= \mathbb{E}\left[e^{-r\tau^*[l]} \max\{v_{\tau^*[l]} - l_{\tau^*[l]}, 0\} - \frac{c}{r}\left(1 - e^{-r\tau^*[l]}\right) | v_0 = x, \tau^*[l] < \delta\right] \mathbb{P}[\tau^*[l] < \delta] \\
&+ \mathbb{E}\left[e^{-r\tau^*[l]} \max\{v_{\tau^*[l]} - l_{\tau^*[l]}, 0\} - \frac{c}{r}\left(1 - e^{-r\tau^*[l]}\right) | v_0 = x, \tau^*[l] \geq \delta\right] \mathbb{P}[\tau^*[l] \geq \delta] + \varepsilon/4 \\
&\leq \mathbb{E}\left[e^{-r\tau^*[l]} \max\{v_{\tau^*[l]} - p_{\tau^*[l]}, 0\} - \frac{c}{r}\left(1 - e^{-r\tau^*[l]}\right) | v_0 = x, \tau^*[l] < \delta\right] \mathbb{P}[\tau^*[l] < \delta] \\
&+ \mathbb{E}\left[e^{-r\tau^*[l]} \max\{v_{\tau^*[l]} - p_{\tau^*[l]}, 0\} - \frac{c}{r}\left(1 - e^{-r\tau^*[l]}\right) | v_0 = x, \tau^*[l] \geq \delta\right] \mathbb{P}[\tau^*[l] \geq \delta] \\
&+ \delta(\varepsilon/4)\mathbb{P}[\tau^*[l] < \delta] + (1-\delta)(\varepsilon/4)\mathbb{P}[\tau^*[l] \geq \delta] + \varepsilon/4 \\
&\leq \mathcal{V}^B(0,x;\tau^*[l], p) + \varepsilon/2
\end{aligned}$$

where the third inequality followed from the optimality of $\tau^*[l]$ under the pricing policy l. □

Proof of Proposition 3. In the special case of linear pricing $t \mapsto p_t := p_0 + \sqrt{\varepsilon}Kt$ the value function takes the form (8) over Ω as we can directly check that it satisfies the PDE of (5). Let's define $K_\pm := \frac{\sqrt{\varepsilon}K \pm \sqrt{\varepsilon K^2 + 2r\sigma^2}}{\sigma^2}$ for convenience. The purchase and exit boundaries ansatz take the form (9). We determine the unknown $A_1, A_2, \bar{V}[\sqrt{\varepsilon}K]$, and $\underline{V}[\sqrt{\varepsilon}K]$ from the boundary conditions

$$V^B(t, \bar{V}_t) = \bar{V}_t - p_t \implies A_1 e^{K_- \bar{V}[\sqrt{\varepsilon}K]} + A_2 e^{K_+ \bar{V}[\sqrt{\varepsilon}K]} - \frac{c}{r} = \bar{V}[\sqrt{\varepsilon}K] \tag{25}$$

$$\partial_x V^B(t,\bar{V}_t) = 1 \implies A_1 K_- e^{K_- \bar{V}[\sqrt{\varepsilon}K]} + A_2 K_+ e^{K_+ \bar{V}[\sqrt{\varepsilon}K]} = 1 \tag{26}$$

$$V^B(t,\underline{V}_t) = 0 \implies A_1 e^{K_- \underline{V}[\sqrt{\varepsilon}K]} + A_2 e^{K_+ \underline{V}[\sqrt{\varepsilon}K]} - \frac{c}{r} = 0 \tag{27}$$

$$\partial_x V^B(t,\bar{V}_t) = 0 \implies A_1 K_- e^{K_- \underline{V}[\sqrt{\varepsilon}K]} + A_2 K_+ e^{K_+ \underline{V}[\sqrt{\varepsilon}K]} = 0 \tag{28}$$

From (27) and (28) we find that

$$A_1 = \frac{c}{r}\left(\frac{K_+}{K_+ - K_-}\right) e^{-K_- \underline{V}[\sqrt{\varepsilon}K]}, \quad A_2 = \frac{c}{r}\left(\frac{K_-}{K_- - K_+}\right) e^{-K_+ \underline{V}[\sqrt{\varepsilon}K]}. \tag{29}$$

Substituting (29) back into (26), we obtain the equation to be solved for $(\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K])$:

$$e^{K_+(\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K])} - e^{K_-(\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K])} = \frac{r}{c} \cdot \frac{K_- - K_+}{K_- K_+}, \tag{30}$$

we note that the LHS is an increasing function, hence the solution always exists. Finally, we find $\bar{V}[\sqrt{\varepsilon}K]$ by substituting (29) back into (25) and simplify:

$$\bar{V}[\sqrt{\varepsilon}K] = \frac{1}{K_-} + \frac{c}{r}\left(e^{K_+(\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K])} - 1\right) \tag{31}$$

from this it is simple to find $\underline{V}[\sqrt{\varepsilon}K]$. Equation (30) and (31) is equivalent to the following non–linear system of equations:

$$\begin{cases} e^{\frac{\sqrt{\varepsilon}K + \sqrt{\varepsilon K^2 + 2r\sigma^2}}{\sigma^2}(\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K])} - e^{\frac{\sqrt{\varepsilon}K - \sqrt{\varepsilon K^2 + 2r\sigma^2}}{\sigma^2}(\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K])} & = \dfrac{\sqrt{\varepsilon K^2 + 2r\sigma^2}}{c} \\ \dfrac{c}{r}\left(e^{\frac{\sqrt{\varepsilon}K + \sqrt{\varepsilon K^2 + 2r\sigma^2}}{\sigma^2}(\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K])} - 1\right) - \bar{V}[\sqrt{\varepsilon}K] & = \dfrac{\sqrt{\varepsilon}K + \sqrt{\varepsilon K^2 + 2r\sigma^2}}{2r} \end{cases} \tag{32}$$

When $\sqrt{\varepsilon}K \sim 0$, we may obtain a simple expression for $\bar{V}[\sqrt{\varepsilon}K]$ and $\underline{V}[\sqrt{\varepsilon}K]$ to the ε–order. We substituting the ansatz (10) into (30), (31), and comparing the zeroth–order and $\sqrt{\varepsilon}$–order terms we get the claimed expression for $\bar{S} := \bar{R}/K, \underline{S} := \underline{R}/K$. The signs of $\bar{S}$ and $\underline{S}$ followed from the Proposition 1, but one can also verify explicitly. □

Proof of Proposition 4. From the first equation of (32), when $c \to 0+$, the RHS becomes large which means $\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K]$ becomes large, and the LHS is $\sim e^{\frac{\sqrt{\varepsilon}K+\sqrt{\varepsilon K^2+2r\sigma^2}}{\sigma^2}(\bar{V}[\sqrt{\varepsilon}K]-\underline{V}[\sqrt{\varepsilon}K])}$. Therefore, the second equation of (32) together with (9) gives:

$$\bar{V}_t = p_0 + \sqrt{\varepsilon}Kt + \frac{\sqrt{\varepsilon K^2 + 2r\sigma^2} - \sqrt{\varepsilon}K}{2r}.$$

and $\underline{V}_t = -\infty$. Therefore, we only have one linearly moving boundary $\bar{V}_t$. Let's assume throughout also that $p_0 \geq g$. The solution $U(t,v)$ to the heat equation with the single linearly moving absorbing boundary with initial condition $U(t=0,v) = \delta(v-x), x \leq \bar{V}_0$, is well–known:

$$U(t,v) = \frac{\exp\left(-\frac{\sqrt{\varepsilon}K}{\sigma^2}(v - x - \sqrt{\varepsilon}Kt) - \frac{\varepsilon K^2}{2\sigma^2}t\right)}{\sigma\sqrt{2\pi t}}\left(e^{-\frac{(v-\sqrt{\varepsilon}Kt-x)^2}{2t\sigma^2}} - e^{-\frac{(v-\sqrt{\varepsilon}Kt+x-2\bar{V}_0)^2}{2t\sigma^2}}\right).$$

Therefore, the purchase probability flux is:

$$-\frac{\sigma^2}{2}\partial_v U(t,\bar{V}_t) = \frac{\bar{V}_0 - x}{\sigma\sqrt{2\pi t^3}}\exp\left(-\frac{\left(\bar{V}_t - x\right)^2}{2t\sigma^2}\right).$$

It is now straightforward to compute the expected firm's payoff at $t=0$:

$$\begin{aligned}\mathcal{V}^S(x;p_0,K) &:= -\frac{\sigma^2}{2}\int_0^\infty e^{-ms}(p_s - g)\partial_v U(s,\bar{V}_s)ds \\ &= \left(p_0 - g + \frac{\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2 + \varepsilon K^2}}\left(p_0 - x + \frac{\sqrt{\varepsilon K^2 + 2r\sigma^2} - \sqrt{\varepsilon}K}{2r}\right)\right) \\ &\quad \times \exp\left(-\left(\frac{\sqrt{\varepsilon}K + \sqrt{2m\sigma^2 + \varepsilon K^2}}{\sigma^2}\right)\left(p_0 - x + \frac{\sqrt{\varepsilon K^2 + 2r\sigma^2} - \sqrt{\varepsilon}K}{2r}\right)\right), \quad (33)\end{aligned}$$

for $x \leq \bar{V}_0$, otherwise if $x > \bar{V}_0$ then we have $\mathcal{V}^S(x;p_0,K) = p_0 - g$. In the special case where

$m = 0$, we have

$$\mathcal{V}^S(x; p_0, K) = \begin{cases} \left(2p_0 - g - x + \dfrac{\sqrt{\varepsilon K^2 + 2r\sigma^2} - \sqrt{\varepsilon}K}{2r}\right) \\ \quad \times \exp\left(-\dfrac{2\sqrt{\varepsilon}K}{\sigma^2}\left(p_0 - x + \dfrac{\sqrt{\varepsilon K^2 + 2r\sigma^2} - \sqrt{\varepsilon}K}{2r}\right)\right), & \sqrt{\varepsilon}K > 0 \\ p_0 - g, & \sqrt{\varepsilon}K = 0 \\ x - g - \dfrac{\sqrt{\varepsilon K^2 + 2r\sigma^2} - \sqrt{\varepsilon}K}{2r}, & \sqrt{\varepsilon}K < 0 \end{cases}.$$

For any fixed p_0, we can approach the supremum $2p_0 + \frac{\sigma}{\sqrt{2r}} - g - x \geq p_0 - g$ of $\mathcal{V}^S$ by choosing $\sqrt{\varepsilon}K \gtrsim 0$ as close to 0 as possible, and earning an extra of $\left(2p_0 + \frac{\sigma}{\sqrt{2r}} - g - x\right) - (p_0 - g) = p_0 - x + \frac{\sigma}{\sqrt{2r}}$. If we can also vary the initial price p, then it is optimal to set p as large as possible, i.e. the optimal price is unbounded.

For $m > 0$, the optimal K is now bounded from 0. This can also be seen for a general p by computing:

$$\frac{\partial \mathcal{V}^S}{\partial K}(x; p_0, K = 0) = e^{-\frac{\sqrt{2m}}{\sigma}\left(p_0 - x + \frac{\sigma}{\sqrt{2r}}\right)} \\ \times \left(\frac{p_0 - x + \sigma/\sqrt{2r}}{\sigma\sqrt{2m}} - (p_0 - g)\left(\frac{p_0 - x + \sigma/\sqrt{2r} - (\sigma/r)\sqrt{m/2}}{\sigma^2}\right)\right),$$

we can see that this is always > 0 for sufficiently small and sufficiently large $m > 0$. $\square$

Proof of Proposition 5. The standard solution U_0 to the heat equation (11) with 2 absorbing non–moving boundaries at $\bar{V}_0 := p_0 + \bar{V}[\sqrt{\varepsilon}K]$, $\underline{V}_0 := p_0 + \underline{V}[\sqrt{\varepsilon}K]$, and the initial condition $U_0(0, v) = \delta(v - x)$ is given by Karatzas and Shreve (2012):

$$U_0(t, v) = \frac{1}{\sigma\sqrt{2\pi t}} \sum_{k=-\infty}^{+\infty} \left[e^{-\frac{(v - x + 2k(\bar{V}_0 - \underline{V}_0))^2}{2t\sigma^2}} - e^{-\frac{(v + x - 2\underline{V}_0 + 2k(\bar{V}_0 - \underline{V}_0))^2}{2t\sigma^2}}\right]. \quad (34)$$

Equivalently:

$$U_0(t,v)dv = \mathbb{P}\left[x+\sigma W_t \in dv, \underline{V}_0 < x+\sigma W_s < \bar{V}_0, s\in[0,t]\right].$$

Instead of moving the boundary according to $\sqrt{\varepsilon}Kt$ we may consider the consumer valuation process to be the Brownian process with drift starting at x: $\tilde{v}_t = x - \sqrt{\varepsilon}Kt + \sigma W_t$ with fixed absorbing boundaries at $\bar{V}_0, \underline{V}_0$. If $\{W_t\}$ is the standard Brownian process on $(\Omega, \mathscr{F}, \Sigma, \mathbb{P})$ then $\{x+\sigma W_t\}$ is the Brownian process with drift starting at x, i.e. $\{\tilde{v}_t\}$ on $(\Omega, \mathscr{F}, \Sigma, \mathbb{Q})$ where

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathscr{F}_t} = \exp\left(-\frac{\sqrt{\varepsilon}K}{\sigma}W_t - \frac{\varepsilon K^2}{2\sigma^2}t\right).$$

Consequently, we have that the solution U to the heat equation (11) with moving boundaries $\bar{V}_t, \underline{V}_t$ is given by

$$\begin{aligned}
U(t,v)dv &= \mathbb{P}\left[\tilde{v}_t \in v-\sqrt{\varepsilon}Kt+dv, \underline{V}_0 < \tilde{v}_s < \bar{V}_0, s\in[0,t]\right] \\
&= \mathbb{Q}\left[x+\sigma W_t \in v-\sqrt{\varepsilon}Kt+dv, \underline{V}_0 < x+\sigma W_s < \bar{V}_0, s\in[0,t]\right] \\
&= \exp\left(-\frac{\sqrt{\varepsilon}K}{\sigma^2}(v-x-\sqrt{\varepsilon}Kt) - \frac{\varepsilon K^2}{2\sigma^2}t\right)U_0(t, v-\sqrt{\varepsilon}Kt)dv
\end{aligned}$$

Therefore, the purchase probability flux is:

$$-\frac{\sigma^2}{2}\partial_v U(t,\bar{V}_t) = \sum_{k=-\infty}^{+\infty}\frac{(2k+1)(\bar{V}_0-\underline{V}_0)-(x-\underline{V}_0)}{\sigma\sqrt{2\pi t^3}}e^{\frac{2\sqrt{\varepsilon}Kk}{\sigma^2}(\bar{V}_0-\underline{V}_0)}e^{-\frac{\left((2k+1)(\bar{V}_0-\underline{V}_0)-(x-\underline{V}_0)+\sqrt{\varepsilon}Kt\right)^2}{2t\sigma^2}}. \tag{35}$$

The term–by–term differentiation is justified at $v=\bar{V}_t$ for any fixed $x\in(\underline{V}_0,\bar{V}_0)$ because $0<|\bar{V}_0-x|<|\bar{V}_0-\underline{V}_0|$, hence the series representation of $U_0(t, v-\sqrt{\varepsilon}Kt)$, and the derivative series both converge absolutely and uniformly for all v in some neighborhoods of $\bar{V}_t$ and $t\in[0,\infty)$. We now compute the seller's expected profit:

Claim 1. *The seller's expected profit from the consumer initially at $x\in(\underline{V}_0,\bar{V}_0)$ is:*

$$
\begin{aligned}
\mathcal{V}^S(x;p_0,K) = {} & \frac{\left(p_0 - g - \frac{\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2+\varepsilon K^2}}(\bar{V}_0 + x - 2\underline{V}_0)\right) e^{-\frac{\sqrt{\varepsilon}K+\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0 - x)}}{1 - e^{-\frac{2\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0-\underline{V}_0)}} \\
& + \frac{\frac{2\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2+\varepsilon K^2}}(\bar{V}_0 - \underline{V}_0)e^{-\frac{\sqrt{\varepsilon}K+\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0 - x)}}{\left(1 - e^{-\frac{2\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0-\underline{V}_0)}\right)^2} \\
& - \frac{\left(p_0 - g - \frac{\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2+\varepsilon K^2}}(\bar{V}_0 - x)\right) e^{-\frac{\sqrt{\varepsilon}K+\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0 - \underline{V}_0)} e^{+\frac{\sqrt{\varepsilon}K-\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(x - \underline{V}_0)}}{1 - e^{-\frac{2\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0-\underline{V}_0)}} \\
& - \frac{\frac{2\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2+\varepsilon K^2}}(\bar{V}_0 - \underline{V}_0)e^{-\frac{\sqrt{\varepsilon}K+\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0 - \underline{V}_0)} e^{+\frac{\sqrt{\varepsilon}K-\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(x - \underline{V}_0)}}{\left(1 - e^{-\frac{2\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0-\underline{V}_0)}\right)^2}, \quad (36)
\end{aligned}
$$

if $m > 0$ or $K \neq 0$, and $\mathcal{V}^S(x;p_0,K) = (p_0 - g)\left(\frac{x-\underline{V}_0}{\bar{V}_0-\underline{V}_0}\right)$ if $m = 0, K = 0$. On the other hand, if $x \leq \underline{V}_0$ then $\mathcal{V}^S(x;p_0,K) = 0$, and if $x \geq \bar{V}_0$ then $\mathcal{V}^s(x;p_0,K) = p_0 - g$.

Proof. We shall only cover the non–trivial case where $x \in (\underline{V}_0, \bar{V}_0)$. First, let's assume that either $m > 0$ or $K \neq 0$. We compute $\mathcal{V}^S(x;p_0,K)$ by substituting (35) into (12):

$$
\begin{aligned}
\mathcal{V}^S(x;p_0,K) &= -\frac{\sigma^2}{2}\int_0^\infty e^{-ms}(p_s - g)\partial_v U(s,\bar{V}_s)ds \\
&= \sum_{k=-\infty}^{+\infty} \left((2k+1)(\bar{V}_0 - \underline{V}_0) - (x - \underline{V}_0)\right) e^{\frac{2\sqrt{\varepsilon}Kk}{\sigma^2}(\bar{V}_0-\underline{V}_0)} \\
&\quad \times \int_0^{+\infty} \frac{(p_0 + \sqrt{\varepsilon}Ks - g)}{\sigma\sqrt{2\pi s^3}} e^{-ms-\frac{\left((2k+1)(\bar{V}_0-\underline{V}_0)-(x-\underline{V}_0)+\sqrt{\varepsilon}Ks\right)^2}{2s\sigma^2}} ds \\
&= \sum_{k=0}^{+\infty}\left(p_0 - g + \frac{\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2+\varepsilon K^2}}\left((2k+1)(\bar{V}_0 - \underline{V}_0) - (x - \underline{V}_0)\right)\right) \\
&\times \exp\left(+\frac{\sqrt{\varepsilon}K}{\sigma^2}\cdot 2k(\bar{V}_0 - \underline{V}_0) - \frac{\sqrt{\varepsilon}K + \sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}\left((2k+1)(\bar{V}_0 - \underline{V}_0) - (x - \underline{V}_0)\right)\right) \\
&- \sum_{k=1}^{+\infty}\left(p_0 - g + \frac{\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2+\varepsilon K^2}}\left((2k-1)(\bar{V}_0 - \underline{V}_0) + (x - \underline{V}_0)\right)\right) \\
&\times \exp\left(-\frac{\sqrt{\varepsilon}K}{\sigma^2}\cdot 2k(\bar{V}_0 - \underline{V}_0) + \frac{\sqrt{\varepsilon}K - \sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}\left((2k-1)(\bar{V}_0 - \underline{V}_0) + (x - \underline{V}_0)\right)\right)
\end{aligned}
$$

In the second equality, we switched the order of summation and integration, which can be justified by Fubini's theorem for $m > 0$ or $K \neq 0$. The resulting infinite series can be evaluated using standard geometric series results to yield (36). If $m = 0$ and $K = 0$, then it is known (see Branco et al. (2012)) that the seller's expected profit is $(p_0 - g)\left(\frac{x - \underline{V}_0}{\bar{V}_0 - \underline{V}_0}\right)$. $\square$

In the limit $\underline{V}_0 \to -\infty$ (i.e. the limit $c \to 0$) (36) reduces to (33) we previously studied. Unlike in the single boundary case, in the presence of the exit boundary, the expected seller's profit is not only continuous at $K = 0$, but also differentiable, even when $m = 0$, as we will show below. We now focus on the $m = 0$ case.

From (36) we have that $\mathcal{V}^S(x; p_0, K < 0)|_{m=0}$ is given by (13), and that:

$$
\begin{aligned}
&\mathcal{V}^S(x; p_0, K > 0)|_{m=0} = \\
&\frac{(p_0 - g - (\bar{V}_0 + x - 2\underline{V}_0)) \exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - x)\right)}{1 - \exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)} + \frac{2(\bar{V}_0 - \underline{V}_0) \exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - x)\right)}{\left(1 - \exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)\right)^2} \\
&- \frac{(p_0 - g - (\bar{V}_0 - x)) \exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)}{1 - \exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)} - \frac{2(\bar{V}_0 - \underline{V}_0) \exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)}{\left(1 - \exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)\right)^2}. \quad (37)
\end{aligned}
$$

Both (13) and (37) are valid expressions for all $K \neq 0$, and with some works, we can show them to be equal for all $K \neq 0$. This proves $\mathcal{V}^S(x; p_0, K)$ is given by (13) for all $K \neq 0$. $\square$

Proof of Lemma 4. We can compute that

$$
\frac{\partial p_0^*}{\partial K}(x, K = 0) = \frac{1}{12 r\sigma}\left(3\sigma - 3\sqrt{\frac{2c^2}{r} + \sigma^2}\sinh^{-1}\sqrt{\frac{r\sigma^2}{2c^2}} - \sigma\left(\sinh^{-1}\sqrt{\frac{r\sigma^2}{2c^2}}\right)^2\right) \leq 0
$$

where the inequality is strict everywhere except at $\sqrt{r}\sigma/c = 0$. Given any q, r, c, σ^2 such that $\sqrt{r}\sigma > 0$ we can find sufficiently small $\lambda_1 > 0$ such that $p_0^*(x, .)$ is a decreasing function for $K \in [-\lambda_1, +\lambda_1]$. If necessary, we can always restrict $\lambda_1 > 0$ to be smaller (i.e. $\lambda_1 < 1$) to ensure that the buyer's respond ε–optimality to $K \in [-\lambda_1, +\lambda_1]$ according to Proposition

3. Any local maximum point of $\mathcal{V}^S(x;.,.)$ would take the form $(p_0^*(x;K^*),K^*)$ where $K^* := \arg\max_K \mathcal{V}^S(x;p_0^*(x,K),K)$, and $p_0^*(x;0)=\hat{p}(x)$, hence $p_0^*<\hat{p}, K^* \gtrsim 0$, or $p_0^*>\hat{p}, K^* \lesssim 0$.

The existence of parameters q,r,c,σ^2 which give examples of maximum point (p_0^*,K^*) satisfying $p_0^*<\hat{p},K^*\gtrsim 0$, or $p_0^*>\hat{p},K^*\lesssim 0$ can be found from Figure 2. In particular, at the boundary between region III and VI (or the boundary between region II and III), $(p_0^*,K^*)=(\hat{p},0)$ is a local maximum, and a unique one if $\lambda_1>0$ is sufficiently small. Due to the continuity of $\mathcal{V}^S(.;.,.)$ and $p_0^*(.,.)$, by choosing a slightly higher q, we obtain an example of a maximum point with $p_0^*>\hat{p},K^*\lesssim 0$, while choosing a slightly lower q, we obtain an example of a maximum point with $p_0^*<\hat{p},K^*\gtrsim 0$. □

Proof of Proposition 6. We first introduce an intuitive technical result.

Lemma 5. *Consider the parameters m,r,c,σ^2, and $\varepsilon>0$, such that $\underline{V}>-\infty$ (i.e. $c>0$). There exists $\bar{x}=\bar{x}(m,r,c,\sigma^2)>0$ such that for all $x\geq g+\bar{x}$, there exists an ε–equilibrium: $(\{\tau^*[p]\in\mathcal{T}\}_{p\in\mathcal{P}_T},p^*\in\mathcal{P}_T)$ such that $\underline{V}_t[p^*]\geq g$, for all $t\in[0,\infty)$.*

Proof. Suppose by contrary that such $\bar{x}>0$ does not exist. Given any $x\in\mathbb{R}$, we can find $p^*\in\mathcal{P}_T$ such that $\mathcal{V}^S(x;\tau^*[p^*],p^*)\geq V^S(x)-\varepsilon$, where $\tau^*[p]\in\mathcal{T}$ satisfies $\mathcal{V}^B(t,x;\tau^*[p],p)\geq V^B(t,x)-\varepsilon$, and let's denote the corresponding exit boundary by $\underline{V}_t[p^*]$. By assumption, we have $\underline{V}_t[p^*]<g$ for some $t\in[0,T]$. But since $\underline{V}[p^*]$ is continuous, for a sufficiently large $\Delta>0$ we must have $\underline{V}_t[p^*]+\Delta>g$ for all $t\in[0,T]$. But then by assumption, $p^*+\Delta\in\mathcal{P}$ cannot be an ε–optimal pricing strategy of the seller when the buyer has an initial valuation $x+\Delta$ because $\underline{V}_t[p^*+\Delta]=\underline{V}_t[p^*]+\Delta>g$ for all t, so there must exist $p^{**}\in\mathcal{P}$ such that $\mathcal{V}^S(x+\Delta;\tau^*[p^{**}],p^{**})>\mathcal{V}^S(x+\Delta;\tau^*[p^*],p^*+\Delta)$. But since $\mathcal{V}^S(x+\Delta;\tau^*[p^*],p^*+\Delta)=\mathcal{V}^S(x;\tau^*[p^*],p^*)+\Delta$, we have that

$$\begin{aligned}\mathcal{V}^S(x;\tau^*[p^{**}],p^{**}-\Delta) &= \mathcal{V}^S(x+\Delta;\tau^*[p^{**}],p^{**})-\Delta\\ &> \mathcal{V}^S(x+\Delta;\tau^*[p^*],p^*+\Delta)-\Delta=\mathcal{V}^S(x;\tau^*[p^*],p^*)\geq V^S(x)-\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary small, we have $\mathcal{V}^S(x; \tau^*[p^{**}], p^{**} - \Delta) \geq V^S(x)$. Relabelling $p^{**} - \Delta$ as p^* and repeating the argument above again we may argue that the last inequality is strict, thus establishing a contradiction. In particular, $p^{**} = p^* + \Delta'$ must be an ε–optimal pricing strategy for the seller, satisfying $\underline{V}_t[p^{**}] > g$ for all $t \in [0, \infty)$, when the buyer has an initial valuation $x + \Delta' > \underline{V}_0[p^*] + \Delta' > g$ for all $\Delta' > \Delta$, and we may define $\bar{x} := x + \Delta - g$. $\square$

Equation (15) in the proposition follows from Proposition 1 and Proposition 2. Since we know that as K increases, the corresponding purchase and exit boundaries $\bar{V}_t[p_0 + Kh]$ and $\underline{V}_t[p_0 + Kh]$ will monotonically decrease and increase toward $p_0 + Kh_t$, respectively. If $x > p_0$ then only the purchase boundary $\bar{V}_0[p_0 + Kh]$ will reach x as $K \to \infty$, giving the seller the payoff $p_0 - g$. Likewise, for $x \leq p_0$ only $\underline{V}_0[p_0 + Kh]$ will reach x as $K \to \infty$ giving the seller the payoff 0.

Let's define $\bar{x} := \bar{x}(m, r, c, \sigma^2)$ as in Lemma 5, then if $x \geq g + \bar{x}$ we can find an ε–optimal strategy $p \in \mathcal{P}_T$ satisfying $\underline{V}_t[p] > g$ for all $t \in [0, \infty)$. It follows that

$$\begin{aligned}\mathcal{V}^S(x; \tau^*[p], p) &= \mathbb{E}\left[e^{-m\tau^*[p]}(p_{\tau^*[p]} - g) \cdot 1_{v_{\tau^*[p]} \geq p_{\tau^*[p]}} \mid v_0 = x\right] \\ &\leq \mathbb{E}\left[\left(p_{\tau^*[p]} - g\right) \cdot 1_{v_{\tau^*[p]} \geq p_{\tau^*[p]}} \mid v_0 = x\right] \leq \mathbb{E}\left[v_{\tau^*[p]} - g \mid v_0 = x\right] = x - g. \quad (38)\end{aligned}$$

The first inequality followed from removing the discounting factor. The second inequality followed by noting that if v_t hits the purchase boundary $\bar{V}_t[p]$ first we would have $v_{\tau^*[p]} - g \geq p_{\tau^*[p]} - g$, and if v_t hits the exit boundary first we would have $v_{\tau^*[p]} < p_{\tau^*[p]}$, so $v_{\tau^*[p]} - g = \underline{V}_{\tau^*[p]}[p] - g \geq 0 = (p_{\tau^*[p]} - g) \cdot 1_{v_{\tau^*[p]} \geq p_{\tau^*[p]}}$. The final equality followed from the Martingale stopping theorem since $\left|v_{t \wedge \tau^*[p]}\right|$ is bounded by $\max_{s \in [0,\infty)}\{|\underline{V}_s[p]|, |\bar{V}_s[p]|\} = \max_{s \in [0,T]}\{|\underline{V}_s[p]|, |\bar{V}_s[p]|\}$ where the latter is finite because both boundaries are continuous over $[0, T]$ and are constant over $[T, \infty)$ by the definition of $\mathcal{P}_T$. So $x - g \geq V^S(x) - \varepsilon$ for any arbitrary $\varepsilon > 0$, hence we conclude that $V^S(x) = x - g$. The claim that this supremum can be approached by (16) follows from (15). $\square$

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